On the Mechanization of the Proof of Hessenberg's Theorem

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Abstract. Hessenberg's Theorem (1905) states that in projective plane geometry Pappus' Axiom implies Desargues' Axiom. Besides being a beautiful theorem, it has an interesting history in that the proof contained a gap for almost 50 years. This makes it worthwhile to formalize the proof by Cronheim (1953), which was claimed to be (and indeed is) complete. The formalization has been carried out in a fragment of first-order logic called coherent logic, with a particularly simple tableaux-like proof procedure. The main novelty here is the use of a tool which has been able to generate large parts of the proof, in particular taking care of the large number of degenerate cases. Our proof is intuitionistic, the Law of Excluded Middle has not been used. All proofs have been independently verified in Coq.

1 Introduction to Projective Plane Geometry

As perceived by the human eye, the two parallel tracks of a railroad seem to meet each other at the horizon. This phenomenon is commonly called *perspectivity* and is caused by the fact that our visual system works through *projection* on the retina. Projective geometry tries to capture this phenomenon (and many others!) by extending the ordinary x, y-plane in the following way.

1. For each direction in the plane an *ideal point* is added serving as the point where all parallel lines of that direction meet.
2. To comply with the axiom that any two points are on a line an *ideal line* is added through all the ideal points.

Ideal points are also called *points at infinity* and likewise the ideal line is called the *line at infinity*. In order to fully comply with the axiom that any two points are on a line, we take the line through a normal point and an ideal point to be the line through the normal point in the direction corresponding to the ideal point. Observe that we now also have achieved that every two lines intersect, including parallel ones and a normal and the ideal line.

The above construction may be visualized as follows. Consider the unit sphere $x^2 + y^2 + z^2 = 1$ and the (tangent) plane $z = 1$. Through projection from the origin, the points in this plane are in one-to-one correspondence with *pairs of antipodal points* on the unit sphere. From the latter we must of course exclude the pairs with $z = 0$. These pairs are a natural choice for points at infinity, since parallel lines in the plane $z = 1$ are projected onto great circles on the unit sphere and the pair of antipodal points with $z = 0$ depends only on the direction. (A great circle on a sphere is the intersection of the sphere with a plane through the center of the sphere. Two parallel lines in $z = 1$ define two planes through the origin which intersect in a third parallel line in $z = 0$.) The great circle with $z = 0$ on the unit sphere finally forms a natural choice for the line at infinity.

The above model, where the 'lines' are great circles on the unit sphere and the 'points' are pairs of antipodal points on the same sphere, may be considered as the standard model of the real projective plane. There are many other projective planes, not necessarily based on the real numbers, even finite ones. In this paper we take the axiomatic approach, where we postulate a number of essential properties shared by all structures that one would like to call 'projective planes'. These may or may not satisfy additional properties, such as those of Pappus and Desargues, whose interrelation we study.

A beautiful principle in projective geometry is that of *duality*. Duality in a projective plane $\mathcal{P}$ means that we can interchange the rôles of points and lines while keeping the incidences. In this way one obtains another projective plane, namely the dual plane $\mathcal{P}^d$. In any statement about incidences which is valid in $\mathcal{P}$, the words *point* and *line* can be systematically interchanged to obtain a dual statement, which is valid in $\mathcal{P}^d$.[3] The duality principle follows from the observation that the axioms either are self-dual (i.e., they imply their own duals), or their duals are themselves listed as axioms. If a statement is valid in all projective planes, then its dual is also valid in all projective planes. Duality is also visible in the standard model:

a *point* is a pair of antipodal points with all great circles through them,
a *line* is a great circle with all pairs of antipodal points on it.

In the next section we identify a fragment of first-order logic in which reasoning is easy and which at the same time is expressive enough to formulate projective geometry.

2 Coherent Logic

As far as we know, Skolem [9] was the first who used coherent logic (*avant la lettre*) to solve a decision problem in lattice theory and to prove the independence of Desargues' Axiom from the basic axioms of projective plane geometry. Modern coherent logic arose in algebraic geometry, see for example [7, Sect. D.1.1]. In this

[3] Duality carries over to derived concepts such as *connecting line, collinear* and *intersection, concurrent*, respectively, to be introduced later.

paper we define coherent logic (abbreviated by CL) as the fragment of first-order logic consisting of implicitly universally quantified implications of the following form:

$$A_1 \wedge \cdots \wedge A_n \rightarrow \exists \boldsymbol{x}_1.\, C_1 \vee \cdots \vee \exists \boldsymbol{x}_m.\, C_m$$

where the A_i are first-order atoms and the C_j are conjunctions of such atoms. We use some obvious notational optimizations to improve readability: if $n = 0$, then we may leave out $\rightarrow$ altogether; if $m = 0$, then we may write $\perp$ (*falsum*) to denote an empty disjunction; empty existential quantifications are left out. Closed atoms, that is, atoms without free variables, are also called *facts*.

CL has a tableaux-like proof theory which is based on using the formulas above as production rules (Skolem: *Erzeugungsprinzipien*) to generate new facts from already known ones, distinguishing cases for each disjunct in the conclusion. Existential quantifiers are eliminated by introducing witnesses. A further explanation of this reasoning mechanism can be found in Section 4. The proof theory is complete and reasoning in CL is constructive in the sense of intuitionistic logic. Based on a simple translation to natural deduction, proofs can be verified directly in the logical framework Coq [10]. All this has been described in [1,2].

In the next section we work towards a machine-oriented axiomatization of projective plane geometry and point out some subtleties w.r.t. the formulation. The complete proof of Hessenberg's Theorem, assembled from three large machine-generated subproofs, is described in Section 5 in a way which is still readable for humans. All files can be found on [3, see `readme`].

3 Axiomatic Projective Plane Geometry

3.1 Axioms for humans

A projective plane is two-sorted: there are *points* and *lines* and there is one primitive relation between these, the *incidence* relation. Let uppercase letters range over points, and lowercase letters over lines. If point P and line l are incident, notation $P\,|\,l$, we say that 'P lies on l' and that 'l passes through P'. A point lying on two lines l and m is called their *intersection*, and is written (lm). Dually, a line passing through two points P and Q is called their *connecting line*, and is written (PQ). A set of points is said to be *collinear* if there exists a line incident with all points in the set. Dually, a set of lines is said to be *concurrent* if there exists a point incident with all lines in the set.

The axioms of projective geometry are as follows.

Axiom 1 *Any two points are incident with a line.*

Dually, we postulate:

Axiom 2 *Any two lines are incident with a point.*

The following self-dual axiom guarantees that (PQ) and (lm) are uniquely determined for distinct P, Q and l, m, respectively.

Axiom 3 *Two distinct points cannot both be incident with two distinct lines.*

There are some subtle differences with other, perhaps more familiar, formulations; we discuss the correspondence with two of the axioms given in [4, p. 13]:

> Any two distinct points (lines) are incident with just one line (point).

It is not hard to see that these statements are implied by Axioms 1–3. For the reverse implication, we need the existence of two distinct points, which follows from other axioms listed in [4].

We now present Pappus' Axiom[4]; see Figure 1. Coxeter [4] states Pappus' Axiom as follows:

> If alternate vertices of a hexagon lie on two lines, the three pairs of opposite sides meet in three collinear points.

Thus, given a hexagon $PQ'RP'QR'$, where the points P, Q, R are on one line and points P', Q', R' are on another line, the pairwise intersections

$$S \equiv ((PQ')(QP')) \qquad T \equiv ((RP')(PR')) \qquad U \equiv ((QR')(RQ'))$$

lie on the so-called *Pappus line*, the dashed line in Figure 1. Following [4], a

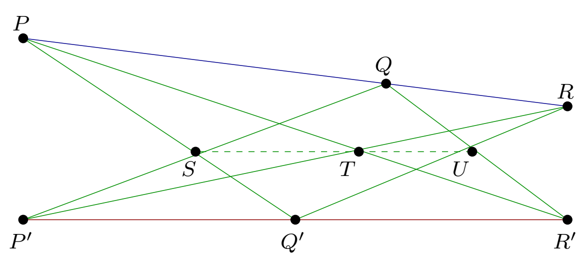

Fig. 1. Pappus' Axiom

Pappus configuration is conveniently given by a 3×3 matrix (P_{ij}). The intersection of $(P_{1i}P_{2j})$ and $(P_{2i}P_{1j})$ is then P_{3k}, for all different i, j, k. (This can be visualized by striking out the rows and columns in which P_{1i}, P_{2j} occur.) For example, the matrix in Figure 1 reads:

$$\begin{pmatrix} P & Q & R \\ P' & Q' & R' \\ U & T & S \end{pmatrix}.$$

[4] Pappus' Axiom is often referred to as Pappus' *Theorem*, because it is true in, for example, the real projective plane. Pappus' Axiom is, however, not true in all projective planes.

In order to exclude some degenerate cases in which the intersections S, T, U are indeterminate and possibly not collinear, Pappus' Axiom requires some side-conditions, of which several alternative formulations (often not explicitly stated) can be found in the literature. We discuss some of these formulations and show how they relate to our formulation.

A. In the formulation of [4], we believe these side-conditions are present in the use of the word 'hexagon'. The hexagon is assumed to be non-degenerate, i.e., its sides are pairwise distinct.
B. A close reading of [5] reveals the assumption of six *distinct* points, three on one line and three on a *distinct* line.
C. Another variation is to require that none of P, Q, R is incident with the line that joins P', Q', R', and vice versa.
D. We choose yet another formulation, which ensures determinacy of the intersections S, T, U by requiring the lines that should determine these intersections to be distinct.

It is easily seen that both A and B imply D. As for C, we formally verified the logical equivalence of the version of Pappus' Axiom with side-conditions C and the one with side-conditions D. The simple fact that C is expressed by a conjunction of length 6, whereas D by one of length 3, seems to lengthen the proof search to such an extent that we prefer the latter:

Axiom 4 (Pappus) *For collinear P, Q, R and collinear P', Q', R', the intersections $((QR')(RQ'))$, $((RP')(PR'))$, and $((PQ')(QP'))$ are collinear if they are determinate.*

As first observed by Hessenberg [6], Axioms 1–4 imply Desargues' Axiom, stated as follows:

Axiom 5 (Desargues) *Two triangles perspective from a point are perspective from a line (under suitable side conditions).*

Two triangles $A_1A_2A_3$ and $B_1B_2B_3$ are said to be perspective from a point S if the three lines joining corresponding vertices meet in S. Dually, two triangles are said to be perspective from a line l if the three intersections of corresponding edges are joined by l. S and l are called the *perspectivity point* and *perspectivity line*, respectively. An example Desargues configuration is depicted in Figure 2.

In the next subsection we present the 'machine' version of Axioms 1–4. This 'mechanization' involves the unfolding of defined notions such as 'collinear', 'determinate' and 'perspective', in terms of the primitive incidence relation. Also, some formulas have to be replaced by (obvious) equivalents in order to comply with the CL-format as defined in the introduction.

3.2 Axioms for machines

Plane geometry with its points and lines is most naturally formalized as a two-sorted theory. There is no principal difficulty in generalizing CL to many-sorted

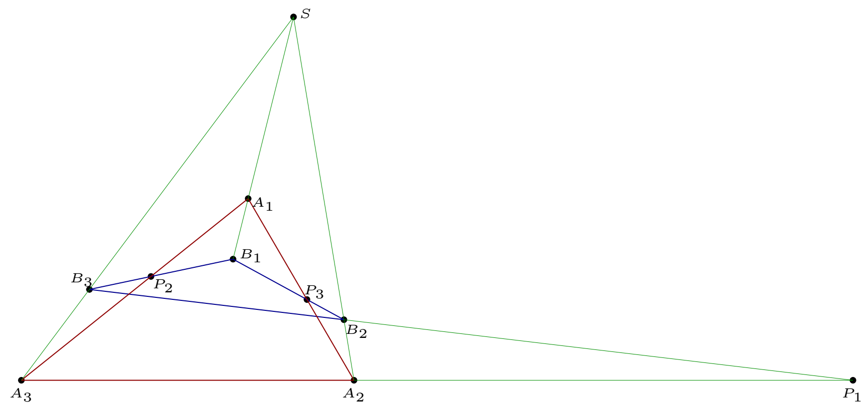

Fig. 2. Example of a Desargues configuration. In space, with the triangles $A_1A_2A_3$ and $B_1B_2B_3$ in different planes, the perspectivity line is the intersection of these two planes. This gives an easy proof for the two-dimensional case as well, essentially using the third dimension.

coherent logic. However, the current implementation supports only one sort. Therefore we have formalized plane geometry as a one-sorted theory (as done in, e.g., [4]).

There exists a standard reduction of many-sorted logic to one-sorted. For every sort s one introduces a unary predicate $s(x)$ whose purpose is to express that x is of sort s. Quantifications with respect to sort s are then relativized. This means that everywhere $\forall_s x.\, \phi$ becomes $\forall x.\, (s(x) \to \phi)$ and $\exists_s x.\, \phi$ becomes $\exists x.\, (s(x) \wedge \phi)$. The equality predicates for the various sorts are all replaced by one-sorted equality $=$. Note that relativization essentially preserves the coherent format. In this way any many-sorted (coherent) formula ϕ can be translated into a one-sorted (coherent) formula ϕ'. We then have the following well-known result:

$$\models_m \phi \text{ iff } \Delta \models \phi' \qquad (*)$$

Here $\models_m$ expresses truth in all many-sorted models and Δ contains some axioms related to the translation $'$. It is standard to let Δ consist of axioms $\exists x.\, s(x)$ for all sorts s. These axioms are necessary, since domains in first-order logic are non-empty, and they are also sufficient for obtaining the equivalence $(*)$. It is less well-known that one may actually add several other axioms to Δ and still get equivalence. Of course the idea is not so much to complicate Δ unnecessarily, but to take advantage of the extra axioms in simplifying ϕ'. We elaborate this in the next paragraphs.

In order to simplify things we restrict attention to the case of plane geometry with sort p for points and l for lines. Besides those two sorts the signature contains $|$ expressing incidence of a point with a line (incidence is not and cannot be a symmetric relation in the many-sorted approach). The one-sorted signature would then consist of two extra unary predicates $p(x)$ and $l(y)$ besides $x \mid y$ expressing that 'object' x lies on 'object' y. Here the intended meaning of 'object' is point for x and line for y, but these meanings are not imposed by one-sorted $\mid$ but by neighbouring atoms $p(x)$ and $l(y)$ coming from the relativized quantifiers.

A more informative $\mid$ is desirable. The axiom we would like to add to the standard Δ is $x \mid y \to p(x) \wedge l(y)$. Why would such an extension be allowed? In order to see this we must enter the standard argument for the equivalence $(*)$, in particular from right to left as strengthening Δ amounts to weakening the right-hand side. This argument is based on transforming any many-sorted model into a one-sorted model of Δ in the following way. Let the one-sorted domain be the union of the domains of the many-sorted model. Interpret the unary predictes $p(x)$ and $l(y)$ as the subsets of points and of lines, respectively, of this union. Interpret $x \mid y$ by the set of pairs of points and lines that are incident in the many-sorted model. The standard argument now proceeds by proving by formula induction that any ϕ is true in the many-sorted model if and only if ϕ is true in the corresponding one-sorted model of Δ. This argument can still be used. The only extra observation we make is that the one-sorted model also validates the axiom $x \mid y \to p(x) \wedge l(y)$ added above. This completes the justification of the extension of Δ.

Thus we add $x \mid y \to p(x) \wedge l(y)$ to the standard axioms:

$$\Delta = \{\exists x.\, p(x), \exists x.\, l(x), (x \mid y \to p(x) \wedge l(y))\}$$

What then are the benefits of this extension? In order to answer this question we observe that $p(x) \wedge \phi \wedge x \mid y$ can be simplified to $\phi \wedge x \mid y$ and $l(y) \wedge \phi \wedge x \mid y$ to $\phi \wedge x \mid y$. This allows us to economize 1, 1, 4 and 18(!) p- and l-atoms in the respective axioms below.

Axiom 6

$$p(x) \wedge p(y) \to \exists u.\, (x \mid u \wedge y \mid u)$$

Axiom 7

$$l(u) \wedge l(v) \to \exists x.\, (x \mid u \wedge x \mid v)$$

Axiom 8

$$x \mid u \wedge x \mid v \wedge y \mid u \wedge y \mid v \to x = y \vee u = v$$

Axiom 9

$$\begin{aligned}
&x_1 \mid u \wedge x_2 \mid u \wedge x_3 \mid u \wedge y_1 \mid v \wedge y_2 \mid v \wedge y_3 \mid v \\
&\wedge x_1 \mid l_1 \wedge y_2 \mid l_1 \wedge p \mid l_1 \wedge x_2 \mid l_2 \wedge y_1 \mid l_2 \wedge p \mid l_2 \\
&\wedge x_1 \mid m_1 \wedge y_3 \mid m_1 \wedge q \mid m_1 \wedge x_3 \mid m_2 \wedge y_1 \mid m_2 \wedge q \mid m_2 \\
&\wedge x_2 \mid n_1 \wedge y_3 \mid n_1 \wedge r \mid n_1 \wedge x_3 \mid n_2 \wedge y_2 \mid n_2 \wedge r \mid n_2 \\
&\to l_1 = l_2 \vee m_1 = m_2 \vee n_1 = n_2 \vee \exists w.\, (p \mid w \wedge q \mid w \wedge r \mid w)
\end{aligned}$$

These axioms in CL format correspond to Axioms 1–4 in the previous section. Some remarks on logical equivalence are in order here. Note the positive formulation of Axiom 8. In Axiom 9, collinearity of x_1, x_2, x_3, that is, $\exists u. (x_1 \mid u \wedge x_2 \mid u \wedge x_3 \mid u)$, has been reformulated using the logical equivalence of $(\exists u. \phi(u)) \to \psi$ and $\forall u. (\phi(u) \to \psi)$ (u not free in ψ). Likewise for the collinearity of y_1, y_2, y_3. The condition enforcing the intersections p, q, r to be determinate, that is, $l_1 \neq l_2$, $m_1 \neq m_2$ and $n_1 \neq n_2$, has been moved to the conclusion using the logical equivalence of $(\neg\phi \wedge \psi) \to \zeta$ and $\psi \to (\phi \vee \zeta)$.

The final step is the mechanization of Desargues' Axiom 5. As this axiom is to be proved as a theorem, we may assume two triangles that are perspective from a point, satisfying certain side conditions. We then have to prove that there exists a perspectivity line. The logical structure of this is extremely simple: a long list of facts (= closed atoms) and negated facts. Finally, the formula to be proven is:

$$\exists l. (p_1 \mid l \wedge p_2 \mid l \wedge p_3 \mid l).$$

This is completely unproblematic from the point of view of CL, but geometrically the situation, in particular with respect to the side conditions, is so complicated that we prefer to explain this in a separate subsection.

3.3 Desargues configurations

Definition 1. *A* Desargues configuration $\mathcal{D}$ *is a sequence of points* $S, A_1, A_2, A_3, B_1, B_2, B_3, P_1, P_2, P_3$ *such that* A_1, A_2, A_3 *are distinct,* B_1, B_2, B_3 *are distinct,* S, A_i, B_i *are collinear for* $i = 1, 2, 3$ *and* (A_jA_k) *and* (B_jB_k) *are distinct lines meeting in* P_i*, for all rotations* (i, j, k) *of* $(1, 2, 3)$.

Observe the following permutation invariance: if we have a Desargues configuration as above, then also $S, A_i, A_j, A_k, B_i, B_j, B_k, P_i, P_j, P_k$ is a Desargues configuration, for any permutation (i, j, k) of $(1, 2, 3)$ (but only rotations will be used). For the purpose of convenient reference, we fix the names of these points, and we let $\mathcal{D}(x, y, z)$ denote the configuration obtained from permuting $(1, 2, 3)$ into (x, y, z). In particular we have $\mathcal{D} = \mathcal{D}(1, 2, 3)$.

Having an automated reasoning tool makes it attractive to experiment with different sets of side conditions. Cronheim's starting point for proving Desargues is a configuration consisting of seven *distinct* points (*non-collinear* A_1, A_2, A_3 and *non-collinear* B_1, B_2, B_3, and a point S) and three *distinct* lines (A_iB_i) which meet in S, the perspectivity point. Note that we have allowed A_1, A_2, A_3 and/or B_1, B_2, B_3 to be collinear but require that corresponding 'edges of the triangles' are distinct.

Our set of conditions in Definition 1 is easily seen to follow from Cronheim's. Assume for example $(A_2A_3) = (B_2B_3)$. Then the points A_2 and B_2 lie on both (A_2B_2) and $(A_2A_3) = (B_2B_3)$. Hence by projective unicity (Axiom 3) the points A_2 and B_2 are equal or we have $(A_2B_2) = (A_2A_3)$. Similarly, A_3 and B_3 are equal or $(A_3B_3) = (A_2A_3)$. In all cases we violate Cronheim's conditions.

For instance, the configurations in Figures 3–5, satisfy our conditions but not Cronheim's. Nevertheless, there is a perspectivity line. It turned out that

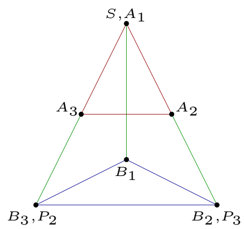

Fig. 3. $A_1 = S$; lines (A_2A_3) and $(B_2B_3) = (P_3P_2)$ meet in P_1 at infinity.

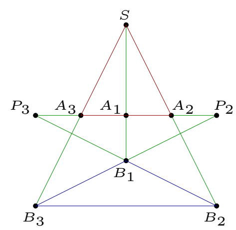

Fig. 4. A degenerate triangle $A_1A_2A_3$, where (A_2A_3) and (B_2B_3) meet in P_1 at infinity.

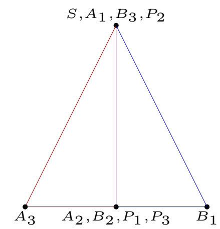

Fig. 5. Another degenerate case, where (A_1A_2) and (B_2B_3) are equal.

the weaker conditions as formulated in Definition 1 were sufficient for the proof.

It is very well possible that the proof can be carried out under even weaker conditions. We have not analyzed this any further. As a historical note we mention that leaving out the condition that A_1, A_2, A_3 are distinct and B_1, B_2, B_3 are distinct, such as done in [9, p. 129], leads to an axiom which implies that *any three points* are collinear. This actually provides an interesting example which is given in full detail in the appendix.

Desargues' Axiom for humans can now be formulated as follows.

Axiom 10 (Desargues) *For any Desargues configuration $\mathcal{D}$ such as in Definition 1, there exists a perspectivity line joining P_1, P_2 and P_3.*

Hessenberg's Theorem for humans states that Axioms 1–4 imply Axiom 10. For machines it reads:

Theorem 11 (Hessenberg). *In the theory consisting of Axioms 6–9, Δ and equality axioms as usual, one can prove:*

$$
\begin{aligned}
& A_1 \mid l_1 \wedge B_1 \mid l_1 \wedge S \mid l_1 \wedge A_2 \mid l_2 \wedge B_2 \mid l_2 \wedge S \mid l_2 \wedge A_3 \mid l_3 \wedge B_3 \mid l_3 \wedge S \mid l_3 \\
& \wedge A_2 \mid a_1 \wedge A_3 \mid a_1 \wedge P_1 \mid a_1 \wedge A_3 \mid a_2 \wedge A_1 \mid a_2 \wedge P_2 \mid a_2 \\
& \wedge A_1 \mid a_3 \wedge A_2 \mid a_3 \wedge P_3 \mid a_3 \wedge B_2 \mid b_1 \wedge B_3 \mid b_1 \wedge P_1 \mid b_1 \\
& \wedge B_3 \mid b_2 \wedge B_1 \mid b_2 \wedge P_2 \mid b_2 \wedge B_1 \mid b_3 \wedge B_2 \mid b_3 \wedge P_3 \mid b_3 \\
\rightarrow\ & A_1 = A_2 \vee A_2 = A_3 \vee A_3 = A_1 \vee B_1 = B_2 \vee B_2 = B_3 \vee B_3 = B_1 \\
& \vee a_1 = b_1 \vee a_2 = b_2 \vee a_3 = b_3 \vee \exists l.\, (P_1 \mid l \wedge P_2 \mid l \wedge P_3 \mid l)
\end{aligned}
$$

4 The Reasoning Mechanism

This section describes how the reasoning mechanism of CL works. Abstractly, this is by forward ground reasoning with case distinction to deal with disjunctions and introduction of new constants to deal with existential quantification.

As an example we prove that any projective plane with at least four points has at least three collinear points. Informally, the proof proceeds as follows.

Consider the intersection Q of the line joining two points with the line joining two other points. If Q is different from the initial four points then both lines have at least three points. Otherwise, if Q is equal to one of the initial four points, say, the first, then the line through the third and the fourth point has at least three points on it.

For a formal proof in CL, consider the theory consisting of Axioms 6,7, Δ and equality axioms. Assume constants P_i and facts axiomatizing them as four different points: $p(P_i)$ and $P_i \neq P_j \quad (1 \leq i < j \leq 4)$, which together form the (initial) *reasoning state*. The goal is to prove:

$$\exists u\, x\, y\, z.\, (x \mid u \wedge y \mid u \wedge z \mid u \wedge x \neq y \wedge y \neq z \wedge x \neq z)$$

For this goal to be a CL formula, $x \neq y$ must be an atomic formula and cannot be taken as shorthand for $x = y \to \bot$. This means that we have to define $\neq$ as the complement of $=$, which is done by extending the theory with the following two CL axioms:

$$x = y \vee x \neq y \qquad x = y \wedge x \neq y \to \bot$$

In the remainder of this section we reason in the extended theory.

Given a reasoning state, a *reasoning step* consists in, first, picking a closed instance $C \to D$ of an axiom which is *invalid* in the state. This means that the antecedent C is true in the state, but the consequent D is not. More precisely, this means that all facts in C occur in the state, but for no disjunct $\exists \boldsymbol{x}.\, C_j$ of D there exist witnesses $\boldsymbol{w}$ such that $C_j[\boldsymbol{x}{:=}\boldsymbol{w}]$ is true in the state. What happens then depends on the form of the conclusion D. If D is a disjunction of length zero, that is, $D = \bot$, then we are done and any conclusion is valid. If D is a disjunction of length one without existential quantifiers, then D is a conjunction of facts and we simply add these facts to the state and continue. If D is a disjunction of length one with existential quantifiers we introduce new constants as witnesses and instantiate D with these constants, add the facts to the state and continue. The state is also understood to be extended by the new constants which may from now on be used in creating new closed instances of axioms.

Let us illustrate the mechanism described so far by elaborating the example. In the initial state above we have the facts $p(P_1)$ and $p(P_2)$. The instance $p(P_1) \wedge p(P_2) \to \exists u.\, (P_1 \mid u \wedge P_2 \mid u)$ of Axiom 6 is invalid since there is no line in the initial state joining P_1 and P_2. Applying this axiom 'remedies' this situation: we add a constant l_{12} and facts $P_1 \mid l_{12}$ and $P_2 \mid l_{12}$ to the state. Note that the name 'l_{12}' is irrelevant as long as it is new. Applying the same axiom, but now instantiated with P_3 and P_4, leads to the further extension of the state with a constant l_{34} and facts $P_3 | l_{34}$ and $P_4 | l_{34}$. The instances $P_1 \,|\, l_{12} \to p(P_1) \wedge l(l_{12})$ and $P_3 \mid l_{34} \to p(P_3) \wedge l(l_{34})$ of the axiom $x \,|\, y \to p(x) \wedge l(y)$ from Δ are invalid, so we add the facts $l(l_{12})$ and $l(l_{34})$ to the state. (The facts $p(P_1)$ and $p(P_3)$ are already present.) Next, we consider the instance $l(l_{12}) \wedge l(l_{34}) \to \exists x.\, (x \mid l_{12} \wedge x \mid l_{34})$ of Axiom 7. This instance is invalid: in the current state there exists no intersection of l_{12} and l_{34}. Therefore we introduce a constant Q and add the facts $Q \mid l_{12}$ and $Q \mid l_{34}$ to the state. In a similar way as above the fact $p(Q)$ is added. Summing up, the reasoning state now extends the initial state with constants l_{12}, l_{34}, Q and facts $l(l_{12}), l(l_{34}), P_1 \mid l_{12}, P_2 \mid l_{12}, P_3 \mid l_{34}, P_4 \mid l_{34}, p(Q), Q \mid l_{12}, Q \mid l_{34}$.

We continue the description of the reasoning mechanism. If D is a disjunction of length greater than one, then the reasoning mechanism distinguishes as many cases as there are disjuncts in the disjunction. These cases are treated as disjunctions of length one as described above. In all these cases the goal has to be proven.

Let us continue the example with a disjunction of length two. In the state we reached above the instance $Q = P_1 \vee Q \neq P_1$ is invalid. Hence we distinguish two cases:

$\underline{Q = P_1}$ We add this fact to the state and infer $P_1 \mid l_{34}$ from $Q \mid l_{34}$. Now that we have the facts $P_1 \mid l_{34}, P_3 \mid l_{34}, P_4 \mid l_{34}, P_1 \neq P_3, P_3 \neq P_4, P_1 \neq P_4$, the goal holds by taking l_{34}, P_1, P_3, P_4 for u, x, y, z, respectively.

$\underline{Q \neq P_1}$ We add this fact to the state and apply the invalid instance $Q = P_2 \vee Q \neq P_2$, which gives rise to two subcases:

- $\underline{Q = P_2}$ This subcase is dealt with analogously to the case $Q = P_1$.
- $\underline{Q \neq P_2}$ We add this fact to the state and have facts $Q \neq P_1, P_1 \neq P_2, Q \neq P_2, Q \mid l_{12}, P_1 \mid l_{12}, P_2 \mid l_{12}$ so that again the goal holds.

In the above argument *almost* all steps have been made explicit. We hope to have convinced the reader of the usefulness of computer support. We have implemented the CL proof procedure in Prolog, see [3, `GL.pl`]. The implementation generates Coq proof objects. The machine has the additional problem that it doesn't know which axiom to apply. The example above can easily be done by brute force. Although in theory the proof of Hessenberg's Theorem can also be found by brute force, in practice one has to specify some 'stepping stones'. About one percent of the total number of reasoning steps had to be specified before the system was able to find the proof.

5 The Complete Proof by Cronheim

Cronheim's proof [5] of Hessenberg's Theorem is three pages long and has a remarkable level of detail. As can be guessed from the length of the machine proof (thousands of steps), there are nevertheless quite a few details left out, something which greatly improves the readability. In some cases, notably Hessenberg's original argument, leaving out 'details' leads to incomplete or wrong proofs. In some rare cases this may even lead to erroneous theorems.

In a formalization these details must all be taken care of, which is time consuming and boring. It is here that we think that tools like the one described in this paper have something to offer. The program turned out to be able to deal with all the details left out by Cronheim. Moreover, we were able to leave out many of the details that Cronheim deemed worth a few lines, mainly the justifications of application of Pappus' Axiom.

Thus the proof scripts are considerably shorter than the original text. The ratio between the proof script describing (or rather generating) the formal proof and the original text in informal mathematics is usually called the *De Bruijn-factor* (after N.G. de Bruijn, see [8]). In the early days of formalization the *De Bruijn-factor* was around 10. Nowadays, it is around 4. Here it is in total around 1, and considerably smaller for some parts of the proof.

We give a high-level exhibition of the machine proof that we have constructed in the proof assistant Coq with the help of a reasoning tool based on CL. The proof closely follows Cronheim's proof. Minor modifications will be justified on the fly.

Cronheim distinguishes two cases: the general case which is caught by Hessenberg's original argument, and a special case which was overseen for fifty years. The case distinction can be phrased as: either ϕ or $\neg\phi$, where ϕ abbreviates: "there exists a permutation (i, j, k) such that $\mathrm{non}(A_i, B_j, B_k)$ and $\mathrm{non}(B_k, A_i, A_j)$".[5] We reformulate $\neg\phi$ into CL by ϕ':

$$\text{for all rotations } (i, j, k) \text{ of } (1, 2, 3),\ A_i \mid (B_j B_k) \text{ or } B_k \mid (A_i A_j) \qquad (\phi')$$

The switch from 'permutation' to 'rotation' will be justified in Subsection 5.2.

Observe that ϕ' amounts to 2^3 cases. In Subsection 5.2, still following Cronheim, it is shown that these can be reduced to 2 cases (Lemma 2): one triangle *circumscribes* the other, a notion defined as follows.

Definition 2. *Triangle $A_1A_2A_3$ circumscribes triangle $B_1B_2B_3$ if for all rotations (i, j, k) of $(1, 2, 3)$ we have $B_i \mid (A_jA_k)$.*

The existence of a perspectivity line in this special setting is proved in Subsection 5.3 (Lemma 3). First, in Subsection 5.1, we treat the $\neg\phi'$-case as proved by Hessenberg's original argument (Lemma 1). Finally, in Subsection 5.4 we assemble these results to prove the main theorem (Theorem 11).

For the CL tool to construct the formal proof of Lemmas 1, 2, and 3, only the essential steps had to be specified, namely the construction of the intersection of two given lines, the construction of a line through two given points and of a new line through *three* given points using Pappus' Axiom. In the latter case it was never necessary to specify the two lines with each three points and the three pairs of lines whose respective intersections are collinear by the new line to be constructed. Neither was it necessary to give the details of the proof that the application of Pappus' Axiom was justified. All files can be found on [3].

5.1 Hessenberg's incomplete argument

In this section we reproduce the argument which Hessenberg took for a complete proof. In fact this argument only proves the existence of a perspectivity line under some additional, non-trivial assumptions.

Lemma 1. *Let $\mathcal{D}$ be a Desargues configuration. Then there exists a perspectivity line joining P_1, P_2, P_3, or $A_1 \mid (B_2B_3)$, or $B_3 \mid (A_1A_2)$.*

[5] The notation (P, Q, R) corresponds to $P \mid (QR)$ and 'non' stands for negation.

Proof. Consider Figure 2 on page 6, and define four further points:

$$\begin{array}{ll} Q \equiv ((A_1A_2)(B_3B_2)) & E \equiv ((A_1A_3)(SQ)) \\ X \equiv ((A_1B_3)(SA_2)) & F \equiv ((B_1B_3)(SQ)) \end{array}$$

Then, the points P_i are shown to be collinear by three applications of Axiom 4, as shown in Figures 6–8.

All files concerning this lemma can be found on [3, cro_case1.*].

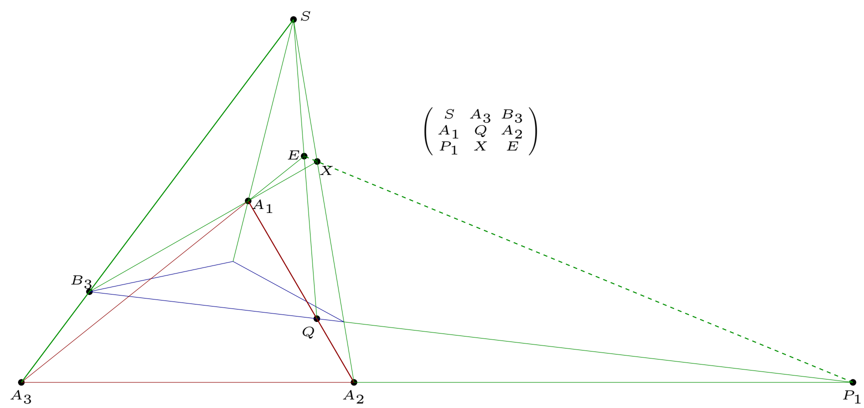

Fig. 6. Proof of Lem. 1: first application of Ax. 4.

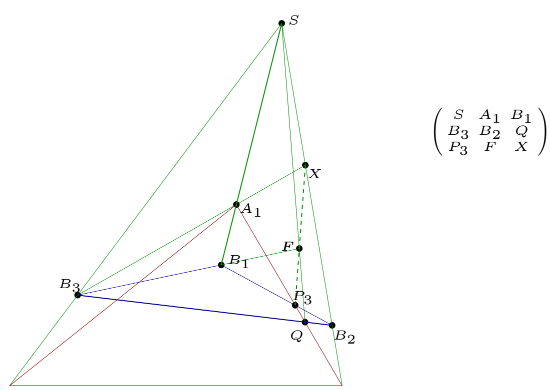

Fig. 7. Proof of Lem. 1: second application of Ax. 4.

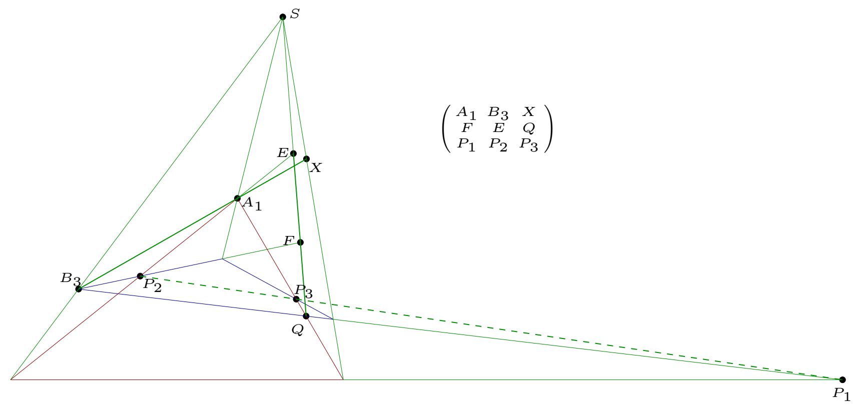

Fig. 8. Proof of Lem. 1: third application of Ax. 4.

The gap in Hessenberg's original proof was that $Q = F = B_3$ if $B_3 \mid (A_1A_2)$ and $Q = E = A_1$ if $A_1 \mid (B_2B_3)$. Then in particular the third application of Pappus' Axiom, see Figure 8, cannot be justified. Therefore the disjuncts $A_1 \mid (B_2B_3)$ and $B_3 \mid (A_1A_2)$ have been added to the conclusion of Lemma 1.

Corollary 1. *Let $\mathcal{D}$ be a Desargues configuration. Then there exists a perspectivity line joining P_1, P_2, P_3 or, for any permutation (i, j, k) of $(1, 2, 3)$, $A_i \mid (B_jB_k)$ or $B_k \mid (A_iA_j)$.*

5.2 Reducing 8 gaps to 2

There is some redundancy in [5, p. 219 (2)] that we wish to avoid in our formalization. Let us first reformulate the premiss of (2) "there does not exist a permutation (i, j, k) of the numbers $(1, 2, 3)$ such that $\text{non}(A_i, B_j, B_k)$ and $\text{non}(B_k, A_i, A_j)$ simultaneously" in a more positive way: for all permutations (i, j, k) one has $A_i \mid (B_jB_k)$ or $B_k \mid (A_iA_j)$. The conclusion of (2) "either (A_x, B_y, B_z) for all permutations (x, y, z) or (B_x, A_y, A_z) for all permutations (x, y, z)" may be paraphrased as: one triangle circumscribes the other. This case is treated in Subsection 5.3.

There are six permutations of the numbers $(1, 2, 3)$. The even permutations correspond to rotations, the odd ones combine rotation with mirroring. In the premiss of (2) an odd permutation boils down to rotating *and interchanging* the two triangles. For example, the rotation $(3, 1, 2)$ yields the disjunction $A_3 \mid (B_1B_2) \vee B_2 \mid (A_3A_1)$. If we interchange the two triangles we get $B_3 \mid (A_1A_2) \vee A_2 \mid (B_3B_1)$, which by commutativity corresponds with $(2, 1, 3)$, indeed an odd permutation. Since the conclusion of (2) is invariant under interchanging the two triangles, it can already be expected that one needs only permutations of one particular sign. This is indeed the case and we prefer to restrict the premiss to the even permutations, that is, to the rotations. Under the premisses of Desargues'

Axiom, which are invariant under interchanging the two triangles, the system can automatically prove Lemma 2. As observed by Cronheim, this lemma is independent of Pappus' Axiom.

Lemma 2. *If* $A_i \mid (B_j B_k) \vee B_k \mid (A_i A_j)$ *for all rotations* (i, j, k) *of* $(1, 2, 3)$*, then there exists a perspectivity line, or triangle* $A_1A_2A_3$ *circumscribes triangle* $B_1B_2B_3$*, that is,* $A_i \mid (B_j B_k)$ *for all rotations* (i, j, k) *of* $(1, 2, 3)$*, or vice versa.*

Adding the disjunct 'there exists a perspectivity line' to the conclusion of the lemma above made it possible to prove Hessenberg's Theorem with weaker side conditions than Cronheim. All files concerning this lemma can be found on [3, cro_8_2.*].

5.3 The special case: one triangle circumscribes the other

An example configuration where triangle $B_1B_2B_3$ circumscribes $A_1A_2A_3$ is depicted in Figure 9.

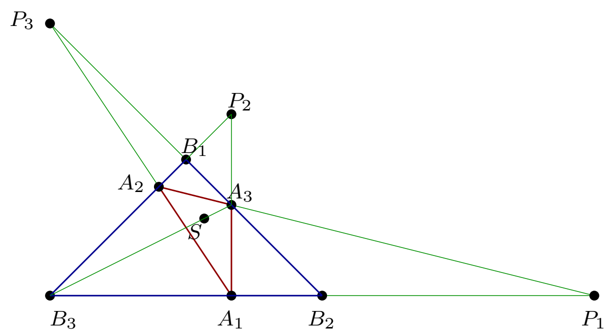

Fig. 9. $B_1B_2B_3$ circumscribes $A_1A_2A_3$.

Lemma 3. *For any Desargues configuration* $\mathcal{D}$ *where triangle* $B_1B_2B_3$ *circumscribes triangle* $A_1A_2A_3$*, there exists a perspectivity line joining the* $P_i = ((A_jA_k)(B_jB_k))$.

Proof. Consider Figure 9, and define the following points:

$$Q_1 \equiv ((B_3P_3)(SA_1)) \qquad Q_2 \equiv ((B_3P_3)(SA_2))$$

Then, collinearity of the P_i follows from three applications of Pappus (Axiom 4), as shown in Figures 10–12, respectively.

All files concerning this lemma can be found on [3, cro_case2.*].

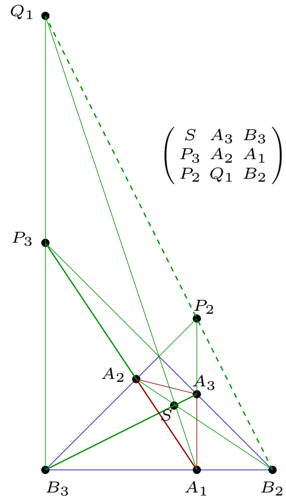

Fig. 10. Proof of Lem. 3: first application of Ax. 4.

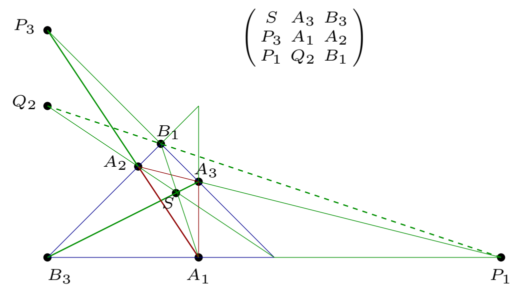

Fig. 11. Proof of Lem. 3: second application of Ax. 4.

5.4 Assembling the parts

We exhibit our formal proof of Hessenberg's Theorem (11), consisting of three applications of Lemma 1, one of Lemma 2 and two of Lemma 3. The corresponding Coq file can be retrieved from [3, ht.v].

Consider a configuration $\mathcal{D}$. The goal, say ψ, is to prove the existence of a perspectivity line joining the points P_i:

$$\psi \equiv \exists l. (P_1 \mid l \wedge P_2 \mid l \wedge P_3 \mid l)$$

Applying Lemma 1 to a configuration $\mathcal{D}(x, y, z)$, we get that either ψ, and then we are done, or $A_x \mid (B_y B_z) \vee B_z \mid (A_x A_y)$. Thus, by application of Lemma 1 to each of the rotations of $(1, 2, 3)$, we have asserted three disjunctions:

$$\begin{gathered} A_1 \mid (B_2 B_3) \vee B_3 \mid (A_1 A_2), \\ A_2 \mid (B_3 B_1) \vee B_1 \mid (A_2 A_3), \\ A_3 \mid (B_1 B_2) \vee B_2 \mid (A_3 A_1). \end{gathered}$$

Given these new facts, we are ready to apply Lemma 2, by which we get two symmetrical cases, either $B_1 B_2 B_3$ circumscribes $A_1 A_2 A_3$, or vice versa. Both cases are solved by application of Lemma 3; for the second case the rôles of A and B in Lemma 3 have to be interchanged.

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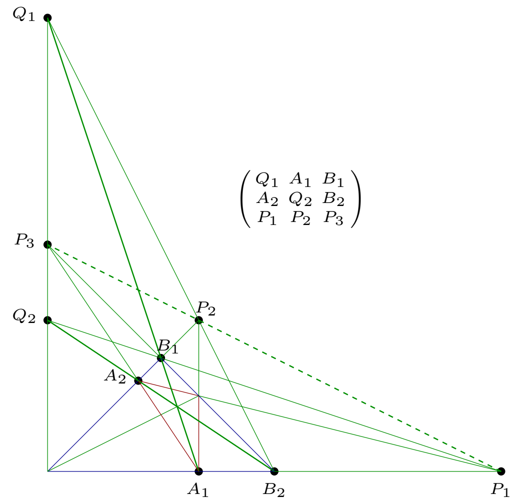

Fig. 12. Proof of Lem. 3: third application of Ax. 4.

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Appendix

In Figure 13 we have reproduced the example (*Beispiel*) from [9, p. 29] in which Skolem illustrates the proof theoretic techniques developed earlier in his paper by showing that Desargues' Axiom is independent of the basic axioms of projective plane geometry. First Skolem formulates Desargues' Axiom as a coherent

> Beispiel: Es sei zu untersuchen, ob der Desarguesche Satz von den homologen Dreiecken aus den aufgestellten Verknüpfungsaxiomen folge oder nicht. Dieser Satz ist ja ein deskriptiver. Er sagt in der kombinatorischen Sprache folgendes: Wenn die Paare
>
> $$\begin{array}{c} (A_1b_1)(A_1c_1)(B_1a_1)(B_1c_1)(C_1a_1)(C_1b_1) \\ (A_2b_2)(A_2c_2)(B_2a_2)(B_2c_2)(C_2a_2)(C_2b_2) \\ (A_1d)(A_2d)(Pd)(B_1e)(B_2e)(Pe)(C_1f)(C_2f)(Pf) \\ (Da_1)(Da_2)(Dp)(Eb_1)(Eb_2)(Ep)(Fc_1)(Fc_2) \end{array} \tag{3}$$
>
> vorkommen, dann soll auch mindenstens eines der Paare
>
> $$(4)\ (Fp)(a_1a_2)(b_1b_2)(c_1c_2)$$
>
> vorhanden sein.

Fig. 13. Fragment from Skolem [9, p. 29].

formula. Instead of using an existential quantifier to express the collinearity of D, E and F he states that any line p joining D and E contains F. Since there is always such a line p, these two formulations are equivalent. More interesting are Skolem's side conditions which appear positively in his disjunctive conclusion $(Fp)(a_1a_2)(b_1b_2)(c_1c_2)$. These side conditions have the same function as the side conditions in Pappus' Axiom, namely to cover the cases in which intersections would become indeterminate. It turns out that Skolem's side conditions are too weak and that therefore his formulation of Desargues' Axiom is too strong. It allows us in fact to prove that any three points are collinear, thus trivializing the projective plane. As this proof is small, it provides a good example to demonstrate our machinery in full detail. By the way, Skolem's proof theoretic argument applies equally well to a correct formulation of Desargues' Axiom, for example, with a disjunctive conclusion

$$(Fp)(a_1a_2)(b_1b_2)(c_1c_2)(A_1B_1)(B_1C_1)(A_1C_1)(A_2B_2)(B_2C_2)(A_2C_2)$$

We would like to stress that we chose this example as a historical anecdote which serves our explanatory purposes well, and that we do not in any way intend to question the value of Skolem's contribution.

The automated reasoning tool [3, `GL.pl`] has been implemented in the programming language Prolog. In the input file below, most of the clauses have the form `<tag> axiom <term> : (<formula>)`. We explain each of the constituents:

`<tag>`, if different from '_', controls the use of the axiom. This is used only for one axiom, tagged `abc(P,Q)`. In combination with the `enabled` and the `next` predicates in the last two clauses of the input this limits the construction of new lines to lines through *different* points `a`, `b`, `c`.

`<term>` gives a name to the axiom including all universally quantified variables. This is used to keep track of which instances of which axioms have been used.

`<formula>` states the coherent formula in question. Here Prolog syntax is used, that is, variables start with a capital, '`,`' stands for conjunction, '`=>`' for implication, '`;`' for disjunction. Finally, `dom` on the right of '`=>`' in the axioms for projective lines and points stands for existential quantification. If `Axiom 3.6` is used, variable `L` is substituted by a fresh object (name), which is subsequently added to the domain. This is the common way of using an existential statement.

We trust that the comments after the symbol '`%`' now sufficiently explain the input file.

```
name('DbyS').                                  % Desargues' Axiom by Skolem

:- dynamic p/1,l/1,i/2,e/2.   % unary predicates: p for point, l for line
                     % binary predicates: i for incidence, e for equality

dom(a). dom(b). dom(c).              % three constants a,b,c in the domain

_ axiom points: (true => p(a),p(b),p(c)).                % a,b,c are points

% goal is proved if a,b,c are collinear
_ axiom goal_proved(L): (i(a,L),i(b,L),i(c,L) => goal).

_ axiom sortp(P,L) : (i(P,L) => p(P)). % incidence pairs have points left
_ axiom sortl(P,L) : (i(P,L) => l(L)). % and lines right

% equality axioms
_ axiom p_ref(X)    : (p(X) => e(X,X)).          % reflexivity for points
_ axiom l_ref(X)    : (l(X) => e(X,X)).          % reflexivity for lines
_ axiom sym(X,Y)    : (e(X,Y) => e(Y,X)).        % symmetry
_ axiom tra(X,Y,Z)  : (e(X,Y),e(Y,Z) => e(X,Z)). % transitivity

% congruence axioms
% equal points lie on the same lines
_ axiom conp(P,Q,L) : (e(P,Q),i(Q,L) => i(P,L)).
% equal lines have the same points
_ axiom conl(P,L,M) : (i(P,L),e(L,M) => i(P,M)).

% projective geometry (Axioms 3.6-3.8)
abc(P,Q) axiom line(P,Q) : (p(P),p(Q) => dom(L),i(P,L),i(Q,L)).
_ axiom point(L,M) : (l(L),l(M) => dom(P),i(P,L),i(P,M)).
```

```
_ axiom unique(P,Q,L,M) : (i(P,L),i(P,M),i(Q,L),i(Q,M) => e(P,Q);e(L,M)).

% Desargues by Skolem
% capital letters L prefix Skolem's names for lines in order to comply
% with Prolog's convention on variables
_ axiom wrong(A1,B1,C1,A2,B2,C2,La1,Lb1,Lc1,La2,Lb2,Lc2,
              P,Ld,Le,Lf,D,E,F,Lp):
(
  i(A1,Lb1),i(A1,Lc1),i(B1,La1),i(B1,Lc1),i(C1,La1),i(C1,Lb1),   % A1B1C1
  i(A2,Lb2),i(A2,Lc2),i(B2,La2),i(B2,Lc2),i(C2,La2),i(C2,Lb2),   % A2B2C2
  i(A1,Ld),i(A2,Ld),i(P,Ld), % \
  i(B1,Le),i(B2,Le),i(P,Le), % - P is point of perspectivity
  i(C1,Lf),i(C2,Lf),i(P,Lf), % /
  % line Lp is the candidate perspectivity line through D and E
  i(D,La1),i(D,La2),i(D,Lp),i(E,Lb1),i(E,Lb2),i(E,Lp),i(F,Lc1),i(F,Lc2)
  =>
  % on which F should lie as well, or a pair of corresp. edges coincides
  i(F,Lp);e(La1,La2);e(Lb1,Lb2);e(Lc1,Lc2)
).

enabled(abc(P,Q),[]) :- member(P,[a,b,c]),member(Q,[a,b,c]),P \= Q.
next(abc(P,Q),[],[]).
```

Next we list the output file, which is self-explaining to a large degree. The only difficult point is the application of Skolem's formulation of Desargues' Axiom, which we will explain at the end.

```
By axiom points using true we have:
  p(a) /\ p(b) /\ p(c)
By axiom p_ref(a) using p(a) we have:
  e(a,a)
By axiom p_ref(b) using p(b) we have:
  e(b,b)
By axiom p_ref(c) using p(c) we have:
  e(c,c)
By axiom line(a,b) using p(a) /\ p(b) we have:
  i(a,w0) /\ i(b,w0)
By axiom sortl(a,w0) using i(a,w0) we have:
  l(w0)
By axiom l_ref(w0) using l(w0) we have:
  e(w0,w0)
By axiom line(a,c) using p(a) /\ p(c) we have:
  i(a,w1) /\ i(c,w1)
By axiom sortl(a,w1) using i(a,w1) we have:
  l(w1)
By axiom l_ref(w1) using l(w1) we have:
  e(w1,w1)
By axiom line(b,c) using p(b) /\ p(c) we have:
  i(b,w2) /\ i(c,w2)
By axiom sortl(b,w2) using i(b,w2) we have:
  l(w2)
```

```
By axiom l_ref(w2) using l(w2) we have:
  e(w2,w2)
By axiom wrong(a,a,a,c,c,a,w0,w0,w0,w1,w1,w2,a,w1,w1,w0,a,a,b,w1)
  using
  i(a,w0) /\ i(a,w0) /\ i(a,w0) /\ i(a,w0) /\ i(a,w0) /\ i(a,w0) /\
  i(c,w1) /\ i(c,w2) /\ i(c,w1) /\ i(c,w2) /\ i(a,w1) /\ i(a,w1) /\
  i(a,w1) /\ i(c,w1) /\ i(a,w1) /\ i(a,w1) /\ i(c,w1) /\ i(a,w1) /\
  i(a,w0) /\ i(a,w0) /\ i(a,w0) /\ i(a,w0) /\ i(a,w1) /\ i(a,w1) /\
  i(a,w0) /\ i(a,w1) /\ i(a,w1) /\ i(b,w0) /\ i(b,w2)
we have:
  i(b,w1) \/ e(w0,w1) \/ e(w0,w1) \/ e(w0,w2)

stack pushed, stack: i(b,w1) \/ e(w0,w1) \/ e(w0,w1) \/ e(w0,w2) :: nil

stack top tailed: i(b,w1)
By axiom goal_proved(w1) using i(a,w1) /\ i(b,w1) /\ i(c,w1) we have:
goal

valid, stack: e(w0,w1) \/ e(w0,w1) \/ e(w0,w2) :: nil

stack top tailed: e(w0,w1)
By axiom sym(w0,w1) using e(w0,w1) we have:
  e(w1,w0)
By axiom conl(b,w0,w1) using i(b,w0) /\ e(w0,w1) we have:
  i(b,w1)
By axiom goal_proved(w1) using i(a,w1) /\ i(b,w1) /\ i(c,w1) we have:
  goal

valid, stack: e(w0,w1) \/ e(w0,w2) :: nil

stack top tailed: e(w0,w1)
By axiom sym(w0,w1) using e(w0,w1) we have:
  e(w1,w0)
By axiom conl(b,w0,w1) using i(b,w0) /\ e(w0,w1) we have:
  i(b,w1)
By axiom goal_proved(w1) using i(a,w1) /\ i(b,w1) /\ i(c,w1) we have:
  goal

valid, stack: e(w0,w2) :: nil

stack popped: e(w0,w2)
By axiom sym(w0,w2) using e(w0,w2) we have:
  e(w2,w0)
By axiom conl(a,w0,w2) using i(a,w0) /\ e(w0,w2) we have:
  i(a,w2)
By axiom goal_proved(w2) using i(a,w2) /\ i(b,w2) /\ i(c,w2) we have:
  goal

valid, stack: nil
```

```
Yes
```

By matching the `wrong`-terms in input and output:

```
wrong(A1,B1,C1,A2,B2,C2,La1,Lb1,Lc1,La2,Lb2,Lc2,P, Ld, Le, Lf,D,E,F,Lp)

wrong( a, a, a, c, c, a, w0, w0, w0, w1, w1, w2,a, w1, w1, w0,a,a,b,w1)
```

we find that Desargues' Axiom in Skolem's formulation is applied by the machine in the following completely degenerated case. The first triangle is the point `a`, the second triangle consists of the points `c`, `a`, with `a` as point of perspectivity. The edges of the first triangle all coincide with the line `w0` joining `a`, `b`. The edges of the second triangle are `w1` joining `a`, `c` (a_2, b_2) and `w2` joining `b`, `c` (c_2). Note that these edges are in accordance with the degeneration of the second triangle. With this particular choice of edges, corresponding edges meet in `a`, `a`, `b`, respectively. Any line through `a` connects `D` and `E`, so in particular `w1` does this. The conclusion that `F` which is equal to `b` lies on `w1` leads to the collinearity of `a`, `b`, `c`, and so does each of the other disjuncts in the conclusion

```
i(b,w1) \/ e(w0,w1) \/ e(w0,w1) \/ e(w0,w2)
```