Metamathematische Methoden in der Geometrie – Metamathematical Methods in Geometry

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with Proofs Generated by an Automated Theorem Prover ArgoCLP
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Contents

Preface

This electronic edition contains all the axioms, definition, and theorems from the book **Metamathematische Methoden in der Geometrie** by Wolfram Schwabhäuser, Wanda Szmielew, and Alfred Tarski (organized in chapters as in the original). All formulae were stored in the TPTP form and for this book translated into a natural language form. The proofs were generated by the ArgoCLP prover (by Sana Stojanović, Vesna Marinković and Predrag Janičić) within the framework described in the paper:

> Sana Stojanović, Julien Narboux and Predrag Janičić: *Synergy Between Interactive and Automated Theorem Proving in Formalization of Mathematical Knowledge: A Case Study of Tarski's Geometry*, 2014.

The resources used for generating this book and Isabelle proofs can be found at `http://argo.matf.bg.ac.rs/?content=downloads`.

Chapter 1

Das Tarskische Axiomensystem, kartesische Räume (Tarski's Axiom System)

Axiom 1 (1) *Have* $(A, B) \cong (B, A)$.

Axiom 2 (2) *Asuming that* $(A, B) \cong (C, D)$, *and* $(A, B) \cong (E, F)$, *it holds that* $(C, D) \cong (E, F)$.

Axiom 3 (3) *Asuming that* $(A, B) \cong (C, C)$, *it holds that* $A = B$.

Axiom 4 (4) *There exist a point* E *such that* $bet(D,A,E)$ *and* $(A, E) \cong (B, C)$.

Axiom 5 (5) *Asuming that* $A \neq B$, $bet(A,B,C)$, $bet(E,F,G)$, $(A, B) \cong (E, F)$, $(B, C) \cong (F, G)$, $(A, D) \cong (E, H)$, *and* $(B, D) \cong (F, H)$, *it holds that* $(C, D) \cong (G, H)$.

Axiom 6 (6) *Asuming that* $bet(A,B,A)$, *it holds that* $A = B$.

Axiom 7 (7) *Asuming that* $bet(A,D,C)$ *and* $bet(B,E,C)$, *it holds that there exist and a point* F *such that* $bet(D,F,B)$ *and* $bet(E,F,A)$.

Axiom 8 (8) *There exist a point* A *and a point* B *and a point* C *such that* $\neg\ bet(A,B,C)$, $\neg\ bet(B,C,A)$ *and* $\neg\ bet(C,A,B)$.

Axiom 9 (9) *Asuming that* $A \neq B$, $(C, A) \cong (C, B)$, $(D, A) \cong (D, B)$, *and* $(E, A) \cong (E, B)$, *it holds that* $bet(C,D,E)$ *or* $bet(D,E,C)$ *or* $bet(E,C,D)$.

Axiom 10 (10) *Asuming that* $bet(A,D,E)$, $bet(B,D,C)$, *and* $A \neq D$, *it holds that there exist a point* F *and a point* G *such that* $bet(A,B,F)$, $bet(A,C,G)$ *and* $bet(F,E,G)$.

Signature of the theory

$col(A, B, C)$	Points A,B,C are colinear
$bet(A, B, C)$	Point B is between points A and C
$AB \cong CD$	Segment AB is congruent to the segment CD
$afs(A, B, C, D, A1, B1, C1, D1)$	`bet(A,B,C) & bet(A1,B1,C1) & cong(A,B,A1,B1) & cong(B,C,B1,C1) & cong(A,D,A1,D1) & cong(B,D,B1,D1)`
$ifs(A, B, C, D, A1, B1, C1, D1)$	`bet(A,B,C) & bet(A1,B1,C1) & cong(A,C,A1,C1) & cong(B,C,B1,C1) & cong(A,D,A1,D1) & cong(C,D,C1,D1)`
$cong3(A, B, C, A1, B1, C1)$	`cong(A,B,A1,B1) & cong(A,C,A1,C1) & cong(B,C,B1,C1)`
$fs(A, B, C, D, A1, B1, C1, D1)$	`col(A,B,C) & cong3(A,B,C,A1,B1,C1) & cong(A,D,A1,D1) & cong(B,D,B1,D1)`
$le(A, B, C, D)$	Segment AB is shorter or equal to segment CD
$lt(A, B, C, D)$	Segment AB is shorter than segment CD
$gt(A, B, C, D)$	Segment AB is longer than segment CD
$ge(A, B, C, D)$	Segment AB is longer or equal to segment CD
$out(A, B, C)$	Point A is outside segment BC

Chapter 2

Folgerungen aus A1 bis A5 (Consequences of the Axioms A1-A5)

Theorem 1 (2.1) *Show that $(A,\, B) \cong (A,\, B)$.*

Proof:

1. It holds that $(B,\, A) \cong (A,\, B)$ (by axiom ax_1).
2. From the facts that $(B,\, A) \cong (A,\, B)$, and $(B,\, A) \cong (A,\, B)$, it holds that $(A,\, B) \cong (A,\, B)$ (by axiom ax_2).

Theorem 2 (2.2) *Assuming that $(A,\, B) \cong (C,\, D)$, show that $(C,\, D) \cong (A,\, B)$.*

Proof:

1. It holds that $(B,\, A) \cong (A,\, B)$ (by axiom ax_1).
2. From the facts that $(B,\, A) \cong (A,\, B)$, and $(B,\, A) \cong (A,\, B)$, it holds that $(A,\, B) \cong (A,\, B)$ (by axiom ax_2).
3. From the facts that $(A,\, B) \cong (C,\, D)$, and $(A,\, B) \cong (A,\, B)$, it holds that $(C,\, D) \cong (A,\, B)$ (by axiom ax_2).

Theorem 3 (2.3) *Assuming that $(A,\, B) \cong (C,\, D)$, and $(C,\, D) \cong (E,\, F)$, show that $(A,\, B) \cong (E,\, F)$.*

Proof:

1. From the facts that $(A,\, B) \cong (C,\, D)$, it holds that $(C,\, D) \cong (A,\, B)$ (by th_2_2).
2. From the facts that $(C,\, D) \cong (A,\, B)$, and $(C,\, D) \cong (E,\, F)$, it holds that $(A,\, B) \cong (E,\, F)$ (by axiom ax_2).

Theorem 4 (2.4) *Assuming that $(A,\, B) \cong (C,\, D)$, show that $(B,\, A) \cong (C,\, D)$.*

Proof:

1. It holds that $(A, B) \cong (B, A)$ (by axiom ax_1).
2. From the facts that $(A, B) \cong (B, A)$, and $(A, B) \cong (C, D)$, it holds that $(B, A) \cong (C, D)$ (by axiom ax_2).

Theorem 5 (2.5) *Assuming that $(A, B) \cong (C, D)$, show that $(A, B) \cong (D, C)$.*

Proof:

1. From the facts that $(A, B) \cong (C, D)$, it holds that $(C, D) \cong (A, B)$ (by th_2_2).
2. From the facts that $(C, D) \cong (A, B)$, it holds that $(D, C) \cong (A, B)$ (by th_2_4).
3. From the facts that $(D, C) \cong (A, B)$, it holds that $(A, B) \cong (D, C)$ (by th_2_2).

Theorem 6 (2.8) *Show that $(A, A) \cong (B, B)$.*

Proof:

1. There exist a point F such that bet(A,A,F) and $(A, F) \cong (B, B)$ (by axiom ax_4).
2. From the facts that $(A, F) \cong (B, B)$, it holds that $A = F$ (by axiom ax_3).
3. From the facts that $(A, F) \cong (B, B)$ and $A = F$ it holds that $(A, A) \cong (B, B)$.

Definition 1 (2.10.1) *Asuming that afs(A,B,C,D,E,F,G,H), it holds that bet(A,B,C), bet(E,F,G), $(A, B) \cong (E, F)$, $(B, C) \cong (F, G)$, $(A, D) \cong (E, H)$ and $(B, D) \cong (F, H)$.*

Definition 2 (2.10.2) *Asuming that bet(A,B,C), bet(E,F,G), $(A, B) \cong (E, F)$, $(B, C) \cong (F, G)$, $(A, D) \cong (E, H)$, and $(B, D) \cong (F, H)$, it holds that afs(A,B,C,D,E,F,G,H).*

Theorem 7 (2.11) *Asuming that bet(A,B,C), bet(D,E,F), $(A, B) \cong (D, E)$, and $(B, C) \cong (E, F)$, it holds that $(A, C) \cong (D, F)$.*

Theorem 8 (2.12) *Asuming that $D \neq A$, bet(D,A,E), $(A, E) \cong (B, C)$, bet(D,A,F), and $(A, F) \cong (B, C)$, it holds that $E = F$.*

Chapter 3

Einfache Sätze über die Zwischenbeziehung (Simple Properties of Betweenness)

Theorem 9 (3.1) *Show that bet(A,B,B).*

Proof:

1. There exist a point G such that bet(A,B,G) and (B, G) $\cong$ (A, A) (by axiom ax_4).
2. From the facts that (B, G) $\cong$ (A, A), it holds that $B = G$ (by axiom ax_3).
3. From the facts that bet(A,B,G) and $B = G$ it holds that bet(A,B,B).

Theorem 10 (3.2) *Assuming that bet(A,B,C), show that bet(C,B,A).*

Proof:

1. It holds that bet(B,C,C) (by th_3_1).
2. From the facts that bet(A,B,C), and bet(B,C,C), there exist a point H such that bet(B,H,B) and bet(C,H,A) (by axiom ax_7).
3. From the facts that bet(B,H,B), it holds that $B = H$ (by axiom ax_6).
4. From the facts that bet(C,H,A) and $B = H$ it holds that bet(C,B,A).

Theorem 11 (3.3) *Show that bet(A,A,B).*

Proof:

1. It holds that bet(B,A,A) (by th_3_1).
2. From the facts that bet(B,A,A), it holds that bet(A,A,B) (by th_3_2).

Theorem 12 (3.4) *Assuming that bet(A,B,C), and bet(B,A,C), show that $A = B$.*

Proof:

Let us prove that $A = B$ by reductio ad absurdum.

1. Assume that $A \neq B$.
2. From the facts that bet(A,B,C), and bet(B,A,C), there exist a point E such that bet(B,E,B) and bet(A,E,A) (by axiom ax_7).
3. From the facts that bet(A,E,A), it holds that $A = E$ (by axiom ax_6).
4. From the facts that bet(B,E,B) and $A = E$ it holds that bet(B,A,B).
5. From the facts that bet(B,A,B), it holds that $B = A$ (by axiom ax_6).
6. From the fact that $A \neq B$, it holds that $B \neq A$ (by the equality axioms).
7. From the facts that $B = A$, and $B \neq A$ we get a contradiction.
   Contradiction.

Therefore, it holds that $A = B$.

This proves the conjecture.

Theorem 13 (3.5) *Asuming that bet(A,B,D) and bet(B,C,D), it holds that bet(A,B,C) and bet(A,C,D).*

Theorem 14 (3.6) *Assuming that bet(A,B,C), and bet(A,C,D), show that bet(B,C,D) and bet(A,B,D).*

Proof:

1. From the facts that bet(A,B,C), it holds that bet(C,B,A) (by th_3_2).
2. From the facts that bet(A,C,D), it holds that bet(D,C,A) (by th_3_2).
3. From the facts that bet(D,C,A), and bet(C,B,A), it holds that bet(D,C,B) and bet(D,B,A) (by th_3_5).
4. From the facts that bet(D,B,A), it holds that bet(A,B,D) (by th_3_2).
5. From the facts that bet(D,C,B), it holds that bet(B,C,D) (by th_3_2).

Theorem 15 (3.7) *Asuming that bet(A,B,C) bet(B,C,D), and $B \neq C$, it holds that bet(A,C,D) and bet(A,B,D).*

Definition 3 (3.8.1) *Asuming that the points A, B, C, D are in that order, it holds that bet(A,B,C), bet(A,B,D), bet(A,C,D) and bet(B,C,D).*

Definition 4 (3.8.2) *Asuming that bet(A,B,C), bet(A,B,D), bet(A,C,D), and bet(B,C,D), it holds that the points A, B, C, D are in that order.*

Theorem 16 (3.9) *Assuming that the points A, B, C, D are in that order, show that the points D, C, B, A are in that order.*

Proof:

1. From the facts that the points A, B, C, D are in that order, it holds that bet(A,B,C), bet(A,B,D), bet(A,C,D) and bet(B,C,D) (by axiom ax_3_8_1).
2. From the facts that bet(A,B,C), it holds that bet(C,B,A) (by th_3_2).
3. From the facts that bet(A,B,D), it holds that bet(D,B,A) (by th_3_2).
4. From the facts that bet(A,C,D), it holds that bet(D,C,A) (by th_3_2).
5. From the facts that bet(B,C,D), it holds that bet(D,C,B) (by th_3_2).
6. From the facts that bet(D,C,B), bet(D,C,A), bet(D,B,A), and bet(C,B,A), it holds that the points D, C, B, A are in that order (by axiom ax_3_8_2).

Theorem 17 (3.10.1) *Assuming that the points A, B, C, D are in that order, show that bet(A,B,C).*

Proof:

1. From the facts that the points A, B, C, D are in that order, it holds that bet(A,B,C), bet(A,B,D), bet(A,C,D) and bet(B,C,D) (by axiom ax_3_8_1).

Theorem 18 (3.10.2) *Assuming that the points A, B, C, D are in that order, show that bet(B,C,D).*

Proof:

1. From the facts that the points A, B, C, D are in that order, it holds that bet(A,B,C), bet(A,B,D), bet(A,C,D) and bet(B,C,D) (by axiom ax_3_8_1).

Theorem 19 (3.11.1) *Assuming that bet(A,B,C), and bet(A,D,B), show that the points A, D, B, C are in that order.*

Proof:

1. From the facts that bet(A,D,B), and bet(A,B,C), it holds that bet(D,B,C) and bet(A,D,C) (by th_3_6).
2. From the facts that bet(A,D,B), bet(A,D,C), bet(A,B,C), and bet(D,B,C), it holds that the points A, D, B, C are in that order (by axiom ax_3_8_2).

Theorem 20 (3.11.2) *Assuming that bet(A,B,C), and bet(B,D,C), show that the points A, B, D, C are in that order.*

Proof:

1. From the facts that bet(A,B,C), and bet(B,D,C), it holds that bet(A,B,D) and bet(A,D,C) (by th_3_5).
2. From the facts that bet(A,B,D), bet(A,B,C), bet(A,D,C), and bet(B,D,C), it holds that the points A, B, D, C are in that order (by axiom ax_3_8_2).

Theorem 21 (3.12.1) *Assuming that bet(A,B,C), bet(B,C,D), and $B \neq C$, show that the points A, B, C, D are in that order.*

Proof:

1. From the facts that bet(A,B,C), bet(B,C,D), and $B \neq C$, it holds that bet(A,C,D) and bet(A,B,D) (by th_3_7).
2. From the facts that bet(A,C,D), and bet(A,B,C), it holds that the points A, B, C, D are in that order (by th_3_11_1).

Theorem 22 (3.12.2) *Assuming that bet(A,B,C), and bet(A,C,D), show that the points A, B, C, D are in that order.*

Proof:

1. From the facts that bet(A,C,D), and bet(A,B,C), it holds that the points A, B, C, D are in that order (by th_3_11_1).

Theorem 23 (3.13) *Show that there exist a point A and a point B such that $A \neq B$.*

Proof:

1. There exist a point A and a point B and a point C such that not bet(A,B,C), not bet(B,C,A) and not bet(C,A,B) (by axiom ax_8).
2. There exist a point D such that bet(A,A,D) and (A, D) $\cong$ (A, A) (by axiom ax_4).
3. It holds that $A = B$ or $A \neq B$ (by axiom ax_g1).
    4. Assume that $A = B$.
        5. It holds that $A = C$ or $A \neq C$ (by axiom ax_g1).
            6. Assume that $A = C$.
                7. It can be trivially proved that $A \neq D$.
                    This proves the conjecture.
            8. Assume that $A \neq C$.
                This proves the conjecture.
    9. Assume that $A \neq B$.
        This proves the conjecture.

Theorem 24 (3.14) *There exist a point C such that bet(A,B,C) and $B \neq C$.*

Theorem 25 (3.15.1) *Assuming that $A \neq B$, show that there exist a point C such that bet(A,B,C) $A \neq C$ and $B \neq C$.*

Proof:

1. There exist a point D such that bet(A,B,D) and $B \neq D$ (by th_3_14).
   Let us prove that $A \neq D$ by reductio ad absurdum.
    2. Assume that $A = D$.
        3. From the facts that bet(A,B,D) and $A = D$ it holds that bet(A,B,A).

4. From the facts that bet(A,B,A), it holds that $A = B$ (by axiom ax_6).
5. From the facts that $A = B$, and $A \neq B$ we get a contradiction.
   Contradiction.

Therefore, it holds that $A \neq D$.

This proves the conjecture.

Theorem 26 (3.15.2) *Asuming that $A \neq B$, it holds that there exist a point C and a point D such that the points A, B, C, D are in that order, $A \neq C$, $A \neq D$, $B \neq C$, $B \neq D$ and $C \neq D$.*

Theorem 27 (3.17) *Asuming that bet(A,B,C), bet(D,E,C), and bet(A,F,D), it holds that there exist a point G such that bet(F,G,C) and bet(B,G,E).*

Chapter 4

Einfache Sätze über Kongruenz und Zwischenbeziehung (Simple Properties of Congruence and Betweenness)

Definition 5 (4.1.1) *Asuming that ifs(A,B,C,D,E,F,G,H), it holds that bet(A,B,C), bet(E,F,G), (A, C) $\cong$ (E, G), (B, C) $\cong$ (F, G), (A, D) $\cong$ (E, H) and (C, D) $\cong$ (G, H).*

Definition 6 (4.1.2) *Asuming that bet(A,B,C), bet(E,F,G), (A, C) $\cong$ (E, G), (B, C) $\cong$ (F, G), (A, D) $\cong$ (E, H), and (C, D) $\cong$ (G, H), it holds that ifs(A,B,C,D,E,F,G,H).*

Theorem 28 (4.2) *Asuming that ifs(A,B,C,D,E,F,G,H), it holds that (B, D) $\cong$ (F, H).*

Theorem 29 (4.3) *Asuming that bet(A,B,C), bet(D,E,F), (A, C) $\cong$ (D, F), and (B, C) $\cong$ (E, F), it holds that (A, B) $\cong$ (D, E).*

Definition 7 (4.4.1) *Asuming that (A, B, C) $\cong$ (D, E, F), it holds that (A, B) $\cong$ (D, E), (A, C) $\cong$ (D, F) and (B, C) $\cong$ (E, F).*

Definition 8 (4.4.2) *Asuming that (A, B) $\cong$ (D, E), (A, C) $\cong$ (D, F), and (B, C) $\cong$ (E, F), it holds that (A, B, C) $\cong$ (D, E, F).*

Definition 9 (4.4.3) *Asuming that (A, B, C,D) is congruent in pairs with (E, F, G,H), it holds that (A, B) $\cong$ (E, F), (A, C) $\cong$ (E, G), (A, D) $\cong$ (E, H), (B, C) $\cong$ (F, G), (B, D) $\cong$ (F, H) and (C, D) $\cong$ (G, H).*

Definition 10 (4.4.4) *Asuming that (A, B) ≅ (E, F), (A, C) ≅ (E, G), (A, D) ≅ (E, H), (B, C) ≅ (F, G), (B, D) ≅ (F, H), and (C, D) ≅ (G, H), it holds that (A, B, C,D) is congruent in pairs with (E, F, G,H).*

Theorem 30 (4.5) *Asuming that bet(A,B,C), and (A, C) ≅ (D, E), it holds that there exist a point F such that bet(D,F,E) and (A, B, C) ≅ (D, F, E).*

Theorem 31 (4.6) *Asuming that bet(A,B,C), and (A, B, C) ≅ (D, E, F), it holds that bet(D,E,F).*

Definition 11 (4.10.1) *Asuming that col(A,B,C), it holds that bet(A,B,C) or bet(B,C,A) or bet(C,A,B).*

Definition 12 (4.10.2) *Asuming that bet(A,B,C), it holds that col(A,B,C).*

Definition 13 (4.10.3) *Asuming that bet(B,C,A), it holds that col(A,B,C).*

Definition 14 (4.10.4) *Asuming that bet(C,A,B), it holds that col(A,B,C).*

Theorem 32 (4.11) *Assuming that col(A,B,C), show that col(B,C,A) col(C,A,B) col(C,B,A) col(B,A,C) and col(A,C,B).*

Proof:

1. From the facts that col(A,B,C), it holds that bet(A,B,C) or bet(B,C,A) or bet(C,A,B) (by axiom ax_4_10_1).
   2. Assume that bet(A,B,C).
      3. From the facts that bet(A,B,C), it holds that col(C,A,B) (by axiom ax_4_10_3).
      4. From the facts that bet(A,B,C), it holds that bet(C,B,A) (by th_3_2).
      5. From the facts that bet(C,B,A), it holds that col(A,C,B) (by axiom ax_4_10_3).
      6. From the facts that bet(C,B,A), it holds that col(C,B,A) (by axiom ax_4_10_2).
      7. From the facts that bet(A,B,C), it holds that col(B,C,A) (by axiom ax_4_10_4).
      8. From the facts that bet(C,B,A), it holds that col(B,A,C) (by axiom ax_4_10_4).
      This proves the conjecture.
   9. Assume that bet(B,C,A).
      10. From the facts that bet(B,C,A), it holds that bet(A,C,B) (by th_3_2).
      11. From the facts that bet(A,C,B), it holds that col(B,A,C) (by axiom ax_4_10_3).
      12. From the facts that bet(A,C,B), it holds that col(A,C,B) (by axiom ax_4_10_2).
      13. From the facts that bet(B,C,A), it holds that col(B,C,A) (by axiom ax_4_10_2).

14. From the facts that bet(A,C,B), it holds that col(C,B,A) (by axiom ax_4_10_4).
15. From the facts that bet(B,C,A), it holds that col(C,A,B) (by axiom ax_4_10_4).
This proves the conjecture.

16. Assume that bet(C,A,B).
17. From the facts that bet(C,A,B), it holds that col(B,C,A) (by axiom ax_4_10_3).
18. From the facts that bet(C,A,B), it holds that bet(B,A,C) (by th_3_2).
19. From the facts that bet(B,A,C), it holds that col(C,B,A) (by axiom ax_4_10_3).
20. From the facts that bet(B,A,C), it holds that col(B,A,C) (by axiom ax_4_10_2).
21. From the facts that bet(C,A,B), it holds that col(C,A,B) (by axiom ax_4_10_2).
22. From the facts that bet(B,A,C), it holds that col(A,C,B) (by axiom ax_4_10_4).
This proves the conjecture.

Theorem 33 (4.12) *Show that col(A,A,B).*

Proof:

1. It holds that bet(A,A,B) (by th_3_3).
2. From the facts that bet(A,A,B), it holds that col(A,A,B) (by axiom ax_4_10_2).

Theorem 34 (4.13) *Assuming that col(A,B,C), and (A, B, C) $\cong$ (D, E, F), show that col(D,E,F).*

Proof:

1. From the facts that (A, B, C) $\cong$ (D, E, F), it holds that (A, B) $\cong$ (D, E), (A, C) $\cong$ (D, F) and (B, C) $\cong$ (E, F) (by axiom ax_4_4_1).
2. From the facts that (A, B) $\cong$ (D, E), it holds that (A, B) $\cong$ (E, D) (by th_2_5).
3. From the facts that (A, B) $\cong$ (E, D), it holds that (B, A) $\cong$ (E, D) (by th_2_4).
4. From the facts that (B, C) $\cong$ (E, F), it holds that (B, C) $\cong$ (F, E) (by th_2_5).
5. From the facts that (B, C) $\cong$ (F, E), it holds that (C, B) $\cong$ (F, E) (by th_2_4).
6. From the facts that (A, C) $\cong$ (D, F), (A, B) $\cong$ (D, E), and (C, B) $\cong$ (F, E), it holds that (A, C, B) $\cong$ (D, F, E) (by axiom ax_4_4_2).
7. From the facts that (B, A) $\cong$ (E, D), (B, C) $\cong$ (E, F), and (A, C) $\cong$ (D, F), it holds that (B, A, C) $\cong$ (E, D, F) (by axiom ax_4_4_2).
8. From the facts that col(A,B,C), it holds that bet(A,B,C) or bet(B,C,A) or bet(C,A,B) (by axiom ax_4_10_1).
9. Assume that bet(A,B,C).
10. From the facts that bet(A,B,C), and (A, B, C) $\cong$ (D, E, F), it holds that bet(D,E,F) (by th_4_6).
11. From the facts that bet(D,E,F), it holds that col(D,E,F) (by axiom ax_4_10_2).
This proves the conjecture.

12. Assume that bet(B,C,A).
13. From the facts that bet(B,C,A), it holds that bet(A,C,B) (by th_3_2).
14. From the facts that bet(A,C,B), and (A, C, B) $\cong$ (D, F, E), it holds that bet(D,F,E) (by th_4_6).
15. From the facts that bet(D,F,E), it holds that col(D,F,E) (by axiom ax_4_10_2).
16. From the facts that col(D,F,E), it holds that col(F,E,D), col(E,D,F), col(E,F,D), col(F,D,E) and col(D,E,F) (by th_4_11).

This proves the conjecture.

17. Assume that bet(C,A,B).
    18. From the facts that bet(C,A,B), it holds that bet(B,A,C) (by th_3_2).
    19. From the facts that bet(B,A,C), and (B, A, C) $\cong$ (E, D, F), it holds that bet(E,D,F) (by th_4_6).
    20. From the facts that bet(E,D,F), it holds that col(E,D,F) (by axiom ax_4_10_2).
    21. From the facts that col(E,D,F), it holds that col(D,F,E), col(F,E,D), col(F,D,E), col(D,E,F) and col(E,F,D) (by th_4_11).

    This proves the conjecture.

Theorem 35 (4.14) *Asuming that col(A,B,C), and (A, B) $\cong$ (D, E), it holds that there exist a point F such that (A, B, C) $\cong$ (D, E, F).*

Definition 15 (4.15.1) *Asuming that fs(A,B,C,D,E,F,G,H), it holds that col(A,B,C), (A, B, C) $\cong$ (E, F, G), (A, D) $\cong$ (E, H) and (B, D) $\cong$ (F, H).*

Definition 16 (4.15.2) *Asuming that col(A,B,C), (A, B, C) $\cong$ (E, F, G), (A, D) $\cong$ (E, H), and (B, D) $\cong$ (F, H), it holds that fs(A,B,C,D,E,F,G,H).*

Theorem 36 (4.16) *Asuming that fs(A,B,C,D,E,F,G,H), and A $\neq$ B, it holds that (C, D) $\cong$ (G, H).*

Theorem 37 (4.17) *Asuming that A $\neq$ B, col(A,B,C), (A, D) $\cong$ (A, E), and (B, D) $\cong$ (B, E), it holds that (C, D) $\cong$ (C, E).*

Theorem 38 (4.18) *Assuming that A $\neq$ B, col(A,B,C), (A, C) $\cong$ (A, D), and (B, C) $\cong$ (B, D), show that C = D.*

Proof:

1. From the facts that $A \neq B$, col(A,B,C), (A, C) $\cong$ (A, D), and (B, C) $\cong$ (B, D), it holds that (C, C) $\cong$ (C, D) (by th_4_17).
2. From the facts that (C, C) $\cong$ (C, D), it holds that (C, D) $\cong$ (C, C) (by th_2_2).
3. From the facts that (C, D) $\cong$ (C, C), it holds that $C = D$ (by axiom ax_3).

Theorem 39 (4.19) *Assuming that bet(A,B,C), (A, B) $\cong$ (A, D), and (C, B) $\cong$ (C, D), show that B = D.*

Proof:

1. It holds that bet(B,A,A) (by th_3_1).
2. From the facts that bet(A,B,C), it holds that col(C,A,B) (by axiom ax_4_10_3).

3. From the facts that $(A,\, B) \cong (A,\, D)$, it holds that $(A,\, D) \cong (A,\, B)$ (by th_2_2).
4. It holds that $A = B$ or $A \neq B$ (by axiom ax_g1).
   5. Assume that $A = B$.
      6. From the facts that $(A,\, D) \cong (A,\, B)$ and $A = B$ it holds that $(A,\, D) \cong (A,\, A)$.
      7. From the facts that $(A,\, D) \cong (A,\, A)$, it holds that $A = D$ (by axiom ax_3).
      This proves the conjecture.
   8. Assume that $A \neq B$.
      Let us prove that $A \neq C$ by reductio ad absurdum.
      9. Assume that $A = C$.
         10. From the facts that bet(A,B,C) and $A = C$ it holds that bet(A,B,A).
         11. From the facts that bet(A,B,A), and bet(B,A,A), it holds that $A = B$ (by th_3_4).
         12. From the facts that $A = B$, and $A \neq B$ we get a contradiction.
            Contradiction.
      Therefore, it holds that $A \neq C$.
      13. From the fact that $A \neq C$, it holds that $C \neq A$ (by the equality axioms).
      14. From the facts that $C \neq A$, col(C,A,B), $(C,\, B) \cong (C,\, D)$, and $(A,\, B) \cong (A,\, D)$, it holds that $B = D$ (by th_4_18).
      This proves the conjecture.

Chapter 5

Konnexität der Zwischenbeziehung und Streckenvergleich (Relationship between Congruence and Comparison of Distances)

Theorem 40 (5.1) *Asuming that $A \neq B$, bet(A,B,C), and bet(A,B,D), it holds that bet(A,C,D) or bet(A,D,C).*

Theorem 41 (5.2) *Assuming that $A \neq B$, bet(A,B,C), and bet(A,B,D), show that bet(B,C,D) or bet(B,D,C).*

Proof:

1. From the facts that $A \neq B$, bet(A,B,C), and bet(A,B,C), it holds that bet(A,C,C) or bet(A,C,C) (by th_5_1).
2. Assume that bet(A,C,C).
3. From the facts that bet(A,B,C), and bet(A,C,C), it holds that bet(B,C,C) and bet(A,B,C) (by th_3_6).
4. From the facts that bet(A,C,C), and bet(A,C,C), it holds that bet(C,C,C) and bet(A,C,C) (by th_3_6).
5. From the facts that $A \neq B$, bet(A,B,C), and bet(A,B,D), it holds that bet(A,C,D) or bet(A,D,C) (by th_5_1).
6. Assume that bet(A,C,D).
7. From the facts that bet(A,B,C), and bet(A,C,D), it holds that bet(B,C,D) and bet(A,B,D) (by th_3_6).

   This proves the conjecture.
8. Assume that bet(A,D,C).
9. From the facts that bet(A,B,D), and bet(A,D,C), it holds that bet(B,D,C) and bet(A,B,C) (by th_3_6).

   This proves the conjecture.
10. Assume that bet(A,C,C).
11. From the facts that bet(A,B,C), and bet(A,C,C), it holds that bet(B,C,C) and bet(A,B,C) (by th_3_6).

12. From the facts that $A \neq B$, bet(A,B,C), and bet(A,B,D), it holds that bet(A,C,D) or bet(A,D,C) (by th_5_1).
   13. Assume that bet(A,C,D).
      14. From the facts that bet(A,B,C), and bet(A,C,D), it holds that bet(B,C,D) and bet(A,B,D) (by th_3_6).
      This proves the conjecture.
   15. Assume that bet(A,D,C).
      16. From the facts that bet(A,B,D), and bet(A,D,C), it holds that bet(B,D,C) and bet(A,B,C) (by th_3_6).
      This proves the conjecture.

Theorem 42 (5.3) *Asuming that bet(A,B,D), and bet(A,C,D), it holds that bet(A,B,C) or bet(A,C,B).*

Definition 17 (5.4.1) *Asuming that le(A,B,C,D), it holds that there exist a point E such that bet(C,E,D) and (A, B) ≅ (C, E).*

Definition 18 (5.4.2) *Asuming that bet(C,E,D), and (A, B) ≅ (C, E), it holds that le(A,B,C,D).*

Definition 19 (5.4.3) *Asuming that ge(C,D,A,B), it holds that le(A,B,C,D).*

Definition 20 (5.4.4) *Asuming that le(A,B,C,D), it holds that ge(C,D,A,B).*

Theorem 43 (5.5.1) *Asuming that le(A,B,C,D), it holds that there exist a point E such that bet(A,B,E) and (A, E) ≅ (C, D).*

Theorem 44 (5.5.2) *Asuming that bet(A,B,E), and (A, E) ≅ (C, D), it holds that le(A,B,C,D).*

Theorem 45 (5.6) *Asuming that le(A,B,C,D), (A, B) ≅ (E, F), and (C, D) ≅ (G, H), it holds that le(E,F,G,H).*

Theorem 46 (5.7) *Show that le(A,B,A,B).*

Proof:

1. It holds that bet(A,B,B) (by th_3_1).
2. It holds that (A, B) $\cong$ (A, B) (by th_2_1).
3. From the facts that bet(A,B,B), and (A, B) $\cong$ (A, B), it holds that le(A,B,A,B) (by axiom ax_5_4_2).

Theorem 47 (5.8) *Asuming that le(A,B,C,D), and le(C,D,E,F), it holds that le(A,B,E,F).*

Theorem 48 (5.9) *Asuming that le(A,B,C,D), and le(C,D,A,B), it holds that (A, B)* $\cong$ *(C, D).*

Theorem 49 (5.10) *Have le(A,B,C,D) or le(C,D,A,B).*

Theorem 50 (5.11) *Show that le(A,A,B,C).*

Proof:

1. It holds that bet(B,B,C) (by th_3_3).
2. It holds that (A, A) $\cong$ (B, B) (by th_2_8).
3. From the facts that bet(B,B,C), and (A, A) $\cong$ (B, B), it holds that le(A,A,B,C) (by axiom ax_5_4_2).

Theorem 51 (5.12.1) *Assuming that col(A,B,C), and bet(A,B,C), show that le(A,B,A,C) and le(B,C,A,C).*

Proof:

4. From the facts that bet(A,B,C), it holds that bet(C,B,A) (by th_3_2).
5. It holds that bet(A,C,C) (by th_3_1).
6. It holds that bet(B,A,A) (by th_3_1).
7. It holds that (A, B) $\cong$ (B, A) (by axiom ax_1).
8. It holds that (B, A) $\cong$ (A, B) (by axiom ax_1).
9. It holds that (B, C) $\cong$ (C, B) (by axiom ax_1).
10. It holds that (C, A) $\cong$ (A, C) (by axiom ax_1).
11. From the facts that bet(A,B,C), and (B, A) $\cong$ (A, B), it holds that le(B,A,A,C) (by axiom ax_5_4_2).
12. From the facts that bet(A,C,C), and (C, A) $\cong$ (A, C), it holds that le(C,A,A,C) (by axiom ax_5_4_2).
13. From the facts that bet(B,A,A), and (A, B) $\cong$ (B, A), it holds that le(A,B,B,A) (by axiom ax_5_4_2).
14. From the facts that bet(C,B,A), and (B, C) $\cong$ (C, B), it holds that le(B,C,C,A) (by axiom ax_5_4_2).
15. From the facts that le(A,B,B,A), and le(B,A,A,C), it holds that le(A,B,A,C) (by th_5_8).
16. From the facts that le(B,C,C,A), and le(C,A,A,C), it holds that le(B,C,A,C) (by th_5_8).

Theorem 52 (5.12.2) *Assuming that col(A,B,C), le(A,B,A,C), and le(B,C,A,C), show that bet(A,B,C).*

Proof:

17. It holds that bet(A,A,B) (by th_3_3).
18. It holds that bet(A,A,C) (by th_3_3).
19. It holds that bet(B,B,A) (by th_3_3).
20. It holds that bet(A,B,B) (by th_3_1).
21. It holds that bet(A,C,C) (by th_3_1).

22. It holds that bet(B,A,A) (by th_3_1).
23. From the facts that bet(A,A,C), it holds that col(A,A,C) (by axiom ax_4_10_2).
24. From the facts that bet(A,C,C), it holds that col(A,C,C) (by axiom ax_4_10_2).
25. From the facts that bet(B,A,A), it holds that col(B,A,A) (by axiom ax_4_10_2).
26. From the facts that bet(B,B,A), it holds that col(B,B,A) (by axiom ax_4_10_2).
27. From the facts that col(A,A,C), and bet(A,A,C), it holds that le(A,A,A,C) and le(A,C,A,C) (by th_5_12_1).
28. From the facts that col(A,C,C), and bet(A,C,C), it holds that le(A,C,A,C) and le(C,C,A,C) (by th_5_12_1).
29. From the facts that col(B,A,A), and bet(B,A,A), it holds that le(B,A,B,A) and le(A,A,B,A) (by th_5_12_1).
30. From the facts that col(B,B,A), and bet(B,B,A), it holds that le(B,B,B,A) and le(B,A,B,A) (by th_5_12_1).
31. From the facts that le(A,C,A,C), and le(A,C,A,C), it holds that $(A,\ C) \cong (A,\ C)$ (by th_5_9).
32. From the facts that le(B,A,B,A), and le(B,A,B,A), it holds that $(B,\ A) \cong (B,\ A)$ (by th_5_9).
33. It holds that $(B,\ A) \cong (A,\ B)$ (by axiom ax_1).
34. From the facts that col(A,B,C), it holds that bet(A,B,C) or bet(B,C,A) or bet(C,A,B) (by axiom ax_4_10_1).
    35. Assume that bet(A,B,C).
        This proves the conjecture.
    36. Assume that bet(B,C,A).
        37. From the facts that bet(B,C,A), it holds that bet(A,C,B) (by th_3_2).
        38. From the facts that bet(A,C,B), it holds that col(A,C,B) (by axiom ax_4_10_2).
        39. From the facts that col(A,C,B), and bet(A,C,B), it holds that le(A,C,A,B) and le(C,B,A,B) (by th_5_12_1).
        40. From the facts that le(A,C,A,B), and le(A,B,A,C), it holds that $(A,\ C) \cong (A,\ B)$ (by th_5_9).
        41. From the facts that $(A,\ C) \cong (A,\ B)$, it holds that $(C,\ A) \cong (A,\ B)$ (by th_2_4).
        42. From the facts that bet(B,C,A), bet(A,A,B), $(B,\ A) \cong (A,\ B)$, and $(C,\ A) \cong (A,\ B)$, it holds that $(B,\ C) \cong (A,\ A)$ (by th_4_3).
        43. From the facts that $(B,\ C) \cong (A,\ A)$, it holds that $B = C$ (by axiom ax_3).
        44. From the facts that bet(A,B,B) and $B = C$ it holds that bet(A,B,C).
        This proves the conjecture.
    45. Assume that bet(C,A,B).
        46. From the facts that bet(C,A,B), it holds that bet(B,A,C) (by th_3_2).
        47. From the facts that bet(B,A,C), it holds that col(B,A,C) (by axiom ax_4_10_2).
        48. From the facts that col(B,A,C), and bet(B,A,C), it holds that le(B,A,B,C) and le(A,C,B,C) (by th_5_12_1).
        49. From the facts that le(A,C,B,C), and le(B,C,A,C), it holds that $(A,\ C) \cong (B,\ C)$ (by th_5_9).
        50. From the facts that bet(A,A,C), bet(B,A,C), $(A,\ C) \cong (B,\ C)$, and $(A,\ C) \cong (A,\ C)$, it holds that $(A,\ A) \cong (B,\ A)$ (by th_4_3).
        51. From the facts that bet(B,A,A), bet(B,B,A), $(B,\ A) \cong (B,\ A)$, and $(A,\ A) \cong (B,\ A)$, it holds that $(B,\ A) \cong (B,\ B)$ (by th_4_3).
        52. From the facts that $(B,\ A) \cong (B,\ B)$, it holds that $B = A$ (by axiom ax_3).
        53. From the facts that bet(A,A,C), $B = A$ and $B = A$ it holds that bet(B,B,C).
        54. From the facts that bet(B,B,C) and $B = A$ it holds that bet(A,B,C).
        This proves the conjecture.

Definition 21 (5.14.1) *Asuming that lt(A,B,C,D), it holds that le(A,B,C,D) and* $(A,\ B) \ncong (C,\ D)$.

Definition 22 (5.14.2) *Asuming that le(A,B,C,D), and* $(A,\ B) \ncong (C,\ D)$, *it holds that lt(A,B,C,D).*

Definition 23 (5.14.3) *Asuming that gt(C,D,A,B), it holds that lt(A,B,C,D).*

Definition 24 (5.14.4) *Asuming that lt(A,B,C,D), it holds that gt(C,D,A,B).*

Chapter 6

Halbgeraden und Geraden (Half-lines and Lines)

Definition 25 (6.1.1) *Asuming that out(C,A,B), it holds that $A \neq C$, $B \neq C$ and bet(C,A,B) or $A \neq C$, $B \neq C$ and bet(C,B,A).*

Definition 26 (6.1.2) *Asuming that $A \neq C$, $B \neq C$, and bet(C,A,B), it holds that out(C,A,B).*

Definition 27 (6.1.3) *Asuming that $A \neq C$, $B \neq C$, and bet(C,B,A), it holds that out(C,A,B).*

Theorem 53 (6.2.1) *Assuming that $A \neq B$, $C \neq B$, $D \neq B$, bet(A,B,D), and bet(C,B,D), show that out(B,A,C).*

Theorem 54 (6.2.2) *Asuming that $A \neq D$, $B \neq D$, $C \neq D$, bet(A,D,C), and out(D,A,B), it holds that bet(B,D,C).*

Theorem 55 (6.3.1) *Asuming that out(C,A,B), it holds that there exist a point D such that $A \neq C$, $B \neq C$, $D \neq C$, bet(A,C,D) and bet(B,C,D).*

Theorem 56 (6.3.2) *Assuming that $A \neq B$, $C \neq B$, $D \neq B$, bet(A,B,D), and bet(C,B,D), show that out(B,A,C).*

Proof:

1. From the facts that $A \neq B$, $C \neq B$, $D \neq B$, bet(A,B,D), and bet(C,B,D), it holds that out(B,A,C) (by th_6_2_1).

Theorem 57 (6.4.1) *Asuming that out(D,A,B), it holds that col(A,D,B) and ¬ bet(A,D,B).*

Theorem 58 (6.4.2) *Assuming that col(A,B,C), and not bet(A,B,C), show that out(B,A,C).*

Proof:

1. It holds that bet(A,B,B) (by th_3_1).
2. It holds that bet(A,A,C) (by th_3_3).
3. From the facts that col(A,B,C), it holds that bet(A,B,C) or bet(B,C,A) or bet(C,A,B) (by axiom ax_4_10_1).
   4. Assume that bet(A,B,C).
      5. From the facts that bet(A,B,C), and not bet(A,B,C) we get a contradiction. Contradiction.
   6. Assume that bet(B,C,A).
      7. It can be trivially proved that $A \neq B$.
         8. It can be trivially proved that $B \neq C$.
            9. From the fact that $B \neq C$, it holds that $C \neq B$ (by the equality axioms).
            10. From the facts that $A \neq B$, $C \neq B$, and bet(B,C,A), it holds that out(B,A,C) (by axiom ax_6_1_3).
                This proves the conjecture.
   11. Assume that bet(C,A,B).
      12. From the facts that bet(C,A,B), it holds that bet(B,A,C) (by th_3_2).
         13. It can be trivially proved that $A \neq B$.
            14. It can be trivially proved that $B \neq C$.
               15. From the fact that $B \neq C$, it holds that $C \neq B$ (by the equality axioms).
               16. From the facts that $A \neq B$, $C \neq B$, and bet(B,A,C), it holds that out(B,A,C) (by axiom ax_6_1_2).
                   This proves the conjecture.

Theorem 59 (6.5) *Assuming that $A \neq B$, show that out(B,A,A).*

Proof:

1. It holds that bet(B,A,A) (by th_3_1).
2. From the facts that $A \neq B$, $A \neq B$, and bet(B,A,A), it holds that out(B,A,A) (by axiom ax_6_1_3).

Theorem 60 (6.6) *Assuming that out(A,B,C), show that out(A,C,B).*

Proof:

1. It holds that $A = B$ or $A \neq B$ (by axiom ax_g1).
   2. Assume that $A = B$.
      3. From the facts that out(A,B,C) and $A = B$ it holds that out(A,A,C).

4. From the facts that out(A,A,C), there exist a point H such that $A \neq A$, $C \neq A$, $H \neq A$, bet(A,A,H) and bet(C,A,H) (by th_6_3_1).
5. From the facts that $A \neq A$, and $A = A$ we get a contradiction.
This proves the conjecture.
6. Assume that $A \neq B$.
Let us prove that $A \neq C$ by reductio ad absurdum.
7. Assume that $A = C$.
8. From the facts that out(A,B,C) and $A = C$ it holds that out(A,B,A).
9. From the facts that out(A,B,A), there exist a point H such that $B \neq A$, $A \neq A$, $H \neq A$, bet(B,A,H) and bet(A,A,H) (by th_6_3_1).
10. From the facts that $A \neq A$, and $A = A$ we get a contradiction.
Contradiction.
Therefore, it holds that $A \neq C$.
11. There exist a point G such that bet(B,A,G) and $A \neq G$ (by th_3_14).
12. From the fact that $A \neq B$, it holds that $B \neq A$ (by the equality axioms).
13. From the fact that $A \neq C$, it holds that $C \neq A$ (by the equality axioms).
14. From the fact that $A \neq G$, it holds that $G \neq A$ (by the equality axioms).
15. From the facts that $B \neq A$, $C \neq A$, $G \neq A$, bet(B,A,G), and out(A,B,C), it holds that bet(C,A,G) (by th_6_2_2).
16. From the fact that $A \neq C$, it holds that $C \neq A$ (by the equality axioms).
17. From the fact that $A \neq B$, it holds that $B \neq A$ (by the equality axioms).
18. From the fact that $A \neq G$, it holds that $G \neq A$ (by the equality axioms).
19. From the facts that $C \neq A$, $B \neq A$, $G \neq A$, bet(C,A,G), and bet(B,A,G), it holds that out(A,C,B) (by th_6_3_2).
This proves the conjecture.

Theorem 61 (6.7) *Assuming that out(A,B,C), and out(A,C,D), show that out(A,B,D).*

Proof:

1. It holds that $A = B$ or $A \neq B$ (by axiom ax_g1).
2. Assume that $A = B$.
3. From the facts that out(A,B,C) and $A = B$ it holds that out(A,A,C).
4. From the facts that out(A,A,C), there exist a point I such that $A \neq A$, $C \neq A$, $I \neq A$, bet(A,A,I) and bet(C,A,I) (by th_6_3_1).
5. From the facts that $A \neq A$, and $A = A$ we get a contradiction.
This proves the conjecture.
6. Assume that $A \neq B$.
7. It holds that $A = C$ or $A \neq C$ (by axiom ax_g1).
8. Assume that $A = C$.
9. From the facts that out(A,C,D) and $A = C$ it holds that out(A,A,D).
10. From the facts that out(A,A,D), there exist a point N such that $A \neq A$, $D \neq A$, $N \neq A$, bet(A,A,N) and bet(D,A,N) (by th_6_3_1).
11. From the facts that $A \neq A$, and $A = A$ we get a contradiction.
This proves the conjecture.

12. Assume that $A \neq C$.
13. It holds that $A = D$ or $A \neq D$ (by axiom ax_g1).
14. Assume that $A = D$.
15. There exist a point H such that bet(B,A,H) and $A \neq H$ (by th_3_14).
16. From the fact that $A \neq B$, it holds that $B \neq A$ (by the equality axioms).
17. From the facts that out(A,C,D) and $A = D$ it holds that out(A,C,A).
18. From the facts that out(A,C,A), there exist a point M such that $C \neq A$, $A \neq A$, $M \neq A$, bet(C,A,M) and bet(A,A,M) (by th_6_3_1).
19. From the facts that $A \neq A$, and $A = A$ we get a contradiction.
This proves the conjecture.
20. Assume that $A \neq D$.
21. There exist a point I such that bet(B,A,I) and $A \neq I$ (by th_3_14).
22. From the fact that $A \neq B$, it holds that $B \neq A$ (by the equality axioms).
23. From the fact that $A \neq C$, it holds that $C \neq A$ (by the equality axioms).
24. From the fact that $A \neq I$, it holds that $I \neq A$ (by the equality axioms).
25. From the facts that $B \neq A$, $C \neq A$, $I \neq A$, bet(B,A,I), and out(A,B,C), it holds that bet(C,A,I) (by th_6_2_2).
26. From the fact that $A \neq C$, it holds that $C \neq A$ (by the equality axioms).
27. From the fact that $A \neq D$, it holds that $D \neq A$ (by the equality axioms).
28. From the fact that $A \neq I$, it holds that $I \neq A$ (by the equality axioms).
29. From the facts that $C \neq A$, $D \neq A$, $I \neq A$, bet(C,A,I), and out(A,C,D), it holds that bet(D,A,I) (by th_6_2_2).
30. From the fact that $A \neq B$, it holds that $B \neq A$ (by the equality axioms).
31. From the fact that $A \neq D$, it holds that $D \neq A$ (by the equality axioms).
32. From the fact that $A \neq I$, it holds that $I \neq A$ (by the equality axioms).
33. From the facts that $B \neq A$, $D \neq A$, $I \neq A$, bet(B,A,I), and bet(D,A,I), it holds that out(A,B,D) (by th_6_3_2).
This proves the conjecture.

Definition 28 (6.8.1) *Asuming that $B \neq A$, and point C is lying on ray (A,B), it holds that out(A,C,B).*

Definition 29 (6.8.2) *Asuming that $B \neq A$, and out(A,C,B), it holds that point C is lying on ray (A,B).*

Theorem 62 (6.11.1) *Asuming that $A \neq B$, and $C \neq D$, it holds that there exist a point E such that out(B,E,A) and (B, E) $\cong$ (C, D).*

Theorem 63 (6.11.2) *Asuming that $A \neq B$, $C \neq D$, out(B,E,A), (B, E) $\cong$ (C, D), out(B,F,A), and (B, F) $\cong$ (C, D), it holds that $E = F$.*

Theorem 64 (6.13.1) *Asuming that out(C,A,B), and le(C,A,C,B), it holds that bet(C,A,B).*

Theorem 65 (6.13.2) *Assuming that out(A,B,C), and bet(A,B,C), show that le(A,B,A,C).*

Proof:

1. From the facts that bet(A,B,C), it holds that col(A,B,C) (by axiom ax_4_10_2).
2. From the facts that col(A,B,C), and bet(A,B,C), it holds that le(A,B,A,C) and le(B,C,A,C) (by th_5_12_1).

Definition 30 (6.14.1) *Asuming that $C \in$ line(A,B), it holds that $A \neq B$ and col(A,B,C).*

Definition 31 (6.14.2) *Asuming that $A \neq B$, and col(A,B,C), it holds that $C \in$ line(A,B).*

Definition 32 (same.lines.1) *Asuming that $A \neq B$, $C \neq D$, $C \in$ line(A,B), and $D \in$ line(A,B), it holds that line(A,B)=line(C,D).*

Definition 33 (same.lines.2) *Asuming that line(A,B)=line(C,D), it holds that $A \neq B$, $C \neq D$, $C \in$ line(A,B) and $D \in$ line(A,B).*

Theorem 66 (6.15.1) *Asuming that $A \neq B$, $A \neq C$, bet(B,A,C), and $D \in$ line(A,B), it holds that point D is lying on ray (A,B) or point D is lying on ray (A,C) or $D = A$.*

Theorem 67 (6.15.2) *Assuming that $A \neq B$, $A \neq C$, bet(B,A,C), and point D is lying on ray (A,B), show that $D \in$ line(A,B).*

Proof:

1. From the fact that $A \neq B$, it holds that $B \neq A$ (by the equality axioms).
2. From the facts that $B \neq A$, and point D is lying on ray (A,B), it holds that out(A,D,B) (by axiom ax_6_8_1).
3. From the facts that out(A,D,B), it holds that col(D,A,B) and not bet(D,A,B) (by th_6_4_1).
4. From the facts that col(D,A,B), it holds that col(A,B,D), col(B,D,A), col(B,A,D), col(A,D,B) and col(D,B,A) (by th_4_11).
5. From the facts that $A \neq B$, and col(A,B,D), it holds that $D \in$ line(A,B) (by axiom ax_6_14_2).

Theorem 68 (6.15.3) *Asuming that $A \neq B$, $A \neq C$, bet(B,A,C), and point D is lying on ray (A,C), it holds that $D \in$ line(A,B).*

Theorem 69 (6.15.4) *Assuming that $A \neq B$, $A \neq C$, bet(B,A,C), and $D = A$, show that $D \in$ line(A,B).*

Proof:

1. It holds that bet(A,A,A) (by th_3_1).
2. It holds that bet(A,B,B) (by th_3_1).
3. It holds that bet(A,C,C) (by th_3_1).
4. It holds that bet(B,A,A) (by th_3_1).
5. It holds that bet(C,A,A) (by th_3_1).
6. From the facts that bet(A,A,A), $D = A$, $D = A$ and $D = A$ it holds that bet(D,D,D).
7. From the facts that bet(D,D,D), $D = A$, $D = A$ and $D = A$ it holds that bet(A,A,A).
8. From the facts that bet(A,B,B) and $D = A$ it holds that bet(D,B,B).
9. From the facts that bet(D,B,B) and $D = A$ it holds that bet(A,B,B).
10. From the facts that bet(A,C,C) and $D = A$ it holds that bet(D,C,C).
11. From the facts that bet(D,C,C) and $D = A$ it holds that bet(A,C,C).
12. From the facts that bet(B,A,A), $D = A$ and $D = A$ it holds that bet(B,D,D).
13. From the facts that bet(B,D,D), $D = A$ and $D = A$ it holds that bet(B,A,A).
14. From the facts that bet(B,A,A), it holds that col(A,B,A) (by axiom ax_4_10_3).
15. From the facts that bet(B,A,C) and $D = A$ it holds that bet(B,D,C).
16. From the facts that bet(B,D,C) and $D = A$ it holds that bet(B,A,C).
17. From the facts that bet(C,A,A), $D = A$ and $D = A$ it holds that bet(C,D,D).
18. From the facts that bet(C,D,D), $D = A$ and $D = A$ it holds that bet(C,A,A).
19. From the facts that $A \neq B$ and $D = A$ it holds that $D \neq B$.
20. From the facts that col(A,B,A), $D = A$ and $D = A$ it holds that col(D,B,D).
21. From the facts that col(D,B,D), $D = A$ and $D = A$ it holds that col(A,B,A).
22. From the facts that $A \neq B$, and col(A,B,A), it holds that $A \in$ line(A,B) (by axiom ax_6_14_2).
23. From the facts that $A \in$ line(A,B), $D = A$ and $D = A$ it holds that $D \in$ line(D,B).
24. From the facts that $D \in$ line(D,B) and $D = A$ it holds that $D \in$ line(A,B).

Theorem 70 (6.16) *Assuming that $A \neq B$, $C \neq A$, and $C \in$ line(A,B), show that line(A,B)=line(A,C).*

Proof:

1. It holds that bet(B,A,A) (by th_3_1).
2. From the facts that bet(B,A,A), it holds that col(A,B,A) (by axiom ax_4_10_3).
3. From the facts that $A \neq B$, and col(A,B,A), it holds that $A \in$ line(A,B) (by axiom ax_6_14_2).
4. From the facts that $A \neq B$, $A \neq C$, $A \in$ line(A,B), and $C \in$ line(A,B), it holds that line(A,B)=line(A,C) (by axiom ax_same_lines_1).

Theorem 71 (6.17) *Assuming that $A \neq B$, show that $A \in$ line(A,B) $B \in$ line(A,B) and line(A,B)=line(B,A).*

Proof:

1. It holds that col(A,A,B) (by th_4_12).
2. It holds that col(B,B,A) (by th_4_12).

3. From the facts that col(A,A,B), it holds that col(A,B,A), col(B,A,A), col(B,A,A), col(A,A,B) and col(A,B,A) (by th_4_11).
4. From the facts that col(B,B,A), it holds that col(B,A,B), col(A,B,B), col(A,B,B), col(B,B,A) and col(B,A,B) (by th_4_11).
5. From the facts that $A \neq B$, and col(A,B,A), it holds that $A \in$ line(A,B) (by axiom ax_6_14_2).
6. From the facts that $A \neq B$, and col(A,B,B), it holds that $B \in$ line(A,B) (by axiom ax_6_14_2).
7. From the fact that $A \neq B$, it holds that $B \neq A$ (by the equality axioms).
8. From the facts that $A \neq B$, $B \neq A$, $B \in$ line(A,B), and $A \in$ line(A,B), it holds that line(A,B)=line(B,A) (by axiom ax_same_lines_1).

Theorem 72 (6.18) *Assuming that $A \neq B$, $C \neq D$, $C \in$ line(A,B), and $D \in$ line(A,B), show that line(A,B)=line(C,D).*

Proof:

1. From the facts that $A \neq B$, $C \neq D$, $C \in$ line(A,B), and $D \in$ line(A,B), it holds that line(A,B)=line(C,D) (by axiom ax_same_lines_1).

Theorem 73 (6.19.1) *Assuming that $A \neq B$, show that there exist a point C and a point D such that $C \neq D$, $A \in$ line(C,D) and $B \in$ line(C,D).*

Proof:

1. From the facts that $A \neq B$, it holds that $A \in$ line(A,B), $B \in$ line(A,B) and line(A,B)=line(B,A) (by th_6_17).

Theorem 74 (6.19.2) *Asuming that $A \neq B$, $C \neq D$, $A \in$ line(C,D), $B \in$ line(C,D), $E \neq F$, $A \in$ line(E,F), and $B \in$ line(E,F), it holds that line(C,D)=line(E,F).*

Theorem 75 (6.21) *Asuming that $A \neq B$, $C \neq D$, line(A,B)$\neq$line(C,D), $E \in$ line(A,B), $E \in$ line(C,D), $F \in$ line(A,B), and $F \in$ line(C,D), it holds that $E = F$.*

Definition 34 (6.22.1) *Asuming that point E is the intersection of line(A,B) and line(C,D), it holds that $A \neq B$, $C \neq D$, $E \in$ line(A,B), $E \in$ line(C,D) and line(A,B)$\neq$line(C,D).*

Definition 35 (6.22.2) *Asuming that $A \neq B$, $C \neq D$, $E \in$ line(A,B), $E \in$ line(C,D), and line(A,B)$\neq$line(C,D), it holds that point E is the intersection of line(A,B) and line(C,D).*

Theorem 76 (6.23.1) *Assuming that col(A,B,C), show that there exist a point D and a point E such that $D \neq E$ $A \in$ line(D,E) $B \in$ line(D,E) and $C \in$ line(D,E).*

Proof:

1. It holds that $A = B$ or $A \neq B$ (by axiom ax_g1).
  2. Assume that $A = B$.
    3. It holds that $A = C$ or $A \neq C$ (by axiom ax_g1).
      4. Assume that $A = C$.
        5. There exist a point D such that bet(A,A,D) and $A \neq D$ (by th_3_14).
        6. From the facts that $A \neq D$, it holds that $A \in$ line(A,D), $D \in$ line(A,D) and line(A,D)=line(D,A) (by th_6_17).
        7. From the facts that $A \in$ line(A,D) and $A = B$ it holds that $B \in$ line(A,D).
        8. From the facts that $A \in$ line(A,D) and $A = C$ it holds that $C \in$ line(A,D).
        This proves the conjecture.
      9. Assume that $A \neq C$.
        10. From the facts that $A \neq C$, it holds that $A \in$ line(A,C), $C \in$ line(A,C) and line(A,C)=line(C,A) (by th_6_17).
        11. From the facts that $A \in$ line(A,C) and $A = B$ it holds that $B \in$ line(A,C).
        This proves the conjecture.
  12. Assume that $A \neq B$.
    13. From the facts that $A \neq B$, it holds that $A \in$ line(A,B), $B \in$ line(A,B) and line(A,B)=line(B,A) (by th_6_17).
    14. From the facts that $A \neq B$, and col(A,B,C), it holds that $C \in$ line(A,B) (by axiom ax_6_14_2).
    This proves the conjecture.

Theorem 77 (6.23.2) *Asuming that $D \neq E$, $A \in$ line(D,E), $B \in$ line(D,E), and $C \in$ line(D,E), it holds that col(A,B,C).*

Theorem 78 (6.24) *Show that there exist a point A and a point B and a point C such that not col(A,B,C).*

Proof: Non-optimized proof.

Theorem 79 (6.25) *Assuming that $A \neq B$, show that there exist a point C such that not col(A,B,C).*

Proof: Non-optimized proof.

Chapter 7

Punktspegelungen (Point reflexivity)

Definition 36 (7.1.1) *Asuming that the point B is midpoint of points A and C, it holds that bet(A,B,C) and $(B,\, A) \cong (B,\, C)$.*

Definition 37 (7.1.2) *Asuming that bet(A,B,C), and $(B,\, A) \cong (B,\, C)$, it holds that the point B is midpoint of points A and C.*

Theorem 80 (7.2) *Assuming that the point A is midpoint of points B and C, show that the point A is midpoint of points C and B.*

Proof:

1. From the facts that the point A is midpoint of points B and C, it holds that bet(B,A,C) and $(A,\, B) \cong (A,\, C)$ (by axiom ax_7_1).
2. From the facts that bet(B,A,C), it holds that bet(C,A,B) (by th_3_2).
3. From the facts that $(A,\, B) \cong (A,\, C)$, it holds that $(A,\, C) \cong (A,\, B)$ (by th_2_2).
4. From the facts that bet(C,A,B), and $(A,\, C) \cong (A,\, B)$, it holds that the point A is midpoint of points C and B (by axiom ax_7_2).

Theorem 81 (7.3.1) *Assuming that the point A is midpoint of points B and B, show that $A = B$.*

Proof:

1. From the facts that the point A is midpoint of points B and B, it holds that bet(B,A,B) and $(A,\, B) \cong (A,\, B)$ (by axiom ax_7_1).
2. From the facts that bet(B,A,B), it holds that $B = A$ (by axiom ax_6).

Theorem 82 (7.3.2) *Show that the point A is midpoint of points A and A.*

Proof:

1. It holds that bet(A,A,A) (by th_3_1).
2. It holds that $(A,\ A) \cong (A,\ A)$ (by th_2_1).
3. From the facts that bet(A,A,A), and $(A,\ A) \cong (A,\ A)$, it holds that the point A is midpoint of points A and A (by axiom ax_7_2).

Theorem 83 (7.4.1) *Show that there exist a point C such that the point A is midpoint of points B and C.*

Proof:

1. There exist a point L such that bet(B,A,L) and $(A,\ L) \cong (A,\ B)$ (by axiom ax_4).
2. From the facts that $(A,\ L) \cong (A,\ B)$, it holds that $(A,\ B) \cong (A,\ L)$ (by th_2_2).
3. From the facts that bet(B,A,L), and $(A,\ B) \cong (A,\ L)$, it holds that the point A is midpoint of points B and L (by axiom ax_7_2).

Theorem 84 (7.4.2) *Assuming that the point A is midpoint of points B and C, and the point A is midpoint of points B and D, show that $C = D$.*

Proof:

1. From the facts that the point A is midpoint of points B and C, it holds that the point A is midpoint of points C and B (by th_7_2).
2. From the facts that the point A is midpoint of points B and D, it holds that the point A is midpoint of points D and B (by th_7_2).
3. From the facts that the point A is midpoint of points B and C, it holds that bet(B,A,C) and $(A,\ B) \cong (A,\ C)$ (by axiom ax_7_1).
4. From the facts that the point A is midpoint of points B and D, it holds that bet(B,A,D) and $(A,\ B) \cong (A,\ D)$ (by axiom ax_7_1).
5. From the facts that the point A is midpoint of points C and B, it holds that bet(C,A,B) and $(A,\ C) \cong (A,\ B)$ (by axiom ax_7_1).
6. From the facts that the point A is midpoint of points D and B, it holds that bet(D,A,B) and $(A,\ D) \cong (A,\ B)$ (by axiom ax_7_1).
7. It holds that $A = B$ or $A \neq B$ (by axiom ax_g1).
   8. Assume that $A = B$.
      9. From the facts that $(A,\ C) \cong (A,\ B)$ and $A = B$ it holds that $(A,\ C) \cong (A,\ A)$.
      10. From the facts that $(A,\ C) \cong (A,\ A)$, it holds that $A = C$ (by axiom ax_3).
      11. From the facts that $(A,\ D) \cong (A,\ B)$ and $A = B$ it holds that $(A,\ D) \cong (A,\ A)$.
      12. From the facts that $(A,\ D) \cong (A,\ A)$, it holds that $A = D$ (by axiom ax_3).
      13. From the facts that $A = B$ and $A = B$ it holds that $A = A$.
      14. From the facts that $A = A$, $A = C$ and $A = D$ it holds that $C = D$.

      This proves the conjecture.
   15. Assume that $A \neq B$.
      16. From the fact that $A \neq B$, it holds that $B \neq A$ (by the equality axioms).
      17. From the facts that $B \neq A$, bet(B,A,C), $(A,\ C) \cong (A,\ B)$, bet(B,A,D), and $(A,\ D) \cong (A,\ B)$, it holds that $C = D$ (by th_2_12).

      This proves the conjecture.

Definition 38 (7.5.1) *Asuming that the point A is symmetry of the point B in relevance to the point C, it holds that the point C is midpoint of points A and B.*

Definition 39 (7.5.2) *Asuming that the point C is midpoint of points A and B, it holds that the point A is symmetry of the point B in relevance to the point C.*

Theorem 85 (7.7) *Assuming that the point A is symmetry of the point B in relevance to the point C, show that the point B is symmetry of the point A in relevance to the point C.*

Proof:

1. From the facts that the point A is symmetry of the point B in relevance to the point C, it holds that the point C is midpoint of points A and B (by axiom ax_7_5_1).
2. From the facts that the point C is midpoint of points A and B, it holds that the point C is midpoint of points B and A (by th_7_2).
3. From the facts that the point C is midpoint of points B and A, it holds that the point B is symmetry of the point A in relevance to the point C (by axiom ax_7_5_2).

Theorem 86 (7.8.1) *Show that there exist a point C such that the point C is symmetry of the point A in relevance to the point B.*

Proof:

1. There exist a point E such that the point B is midpoint of points A and E (by th_7_4_1).
2. From the facts that the point B is midpoint of points A and E, it holds that the point B is midpoint of points E and A (by th_7_2).
3. From the facts that the point B is midpoint of points E and A, it holds that the point E is symmetry of the point A in relevance to the point B (by axiom ax_7_5_2).

Theorem 87 (7.8.2) *Assuming that the point A is symmetry of the point B in relevance to the point C, and the point D is symmetry of the point B in relevance to the point C, show that $A = D$.*

Proof:

1. From the facts that the point A is symmetry of the point B in relevance to the point C, it holds that the point C is midpoint of points A and B (by axiom ax_7_5_1).
2. From the facts that the point C is midpoint of points A and B, it holds that the point C is midpoint of points B and A (by th_7_2).
3. From the facts that the point D is symmetry of the point B in relevance to the point C, it holds that the point C is midpoint of points D and B (by axiom ax_7_5_1).
4. From the facts that the point C is midpoint of points D and B, it holds that the point C is midpoint of points B and D (by th_7_2).

5. From the facts that the point C is midpoint of points B and A, and the point C is midpoint of points B and D, it holds that $A = D$ (by th_7_4_2).

Theorem 88 (7.10.1) *Assuming that the point A is symmetry of the point A in relevance to the point B, show that $A = B$.*

Proof:

1. From the facts that the point A is symmetry of the point A in relevance to the point B, it holds that the point B is midpoint of points A and A (by axiom ax_7_5_1).
2. From the facts that the point B is midpoint of points A and A, it holds that $B = A$ (by th_7_3_1).

Theorem 89 (7.10.2) *Show that the point A is symmetry of the point A in relevance to the point A.*

Proof:

1. It holds that the point A is midpoint of points A and A (by th_7_3_2).
2. From the facts that the point A is midpoint of points A and A, it holds that the point A is symmetry of the point A in relevance to the point A (by axiom ax_7_5_2).

Theorem 90 (7.13) *Asuming that the point A is symmetry of the point D in relevance to the point C, and the point B is symmetry of the point E in relevance to the point C, it holds that $(A, B) \cong (D, E)$.*

Theorem 91 (7.15.1) *Assuming that bet(A,B,C), the point A is symmetry of the point D in relevance to the point E, the point B is symmetry of the point F in relevance to the point E, and the point C is symmetry of the point G in relevance to the point E, show that bet(D,F,G).*

Proof:

1. From the facts that the point A is symmetry of the point D in relevance to the point E, and the point B is symmetry of the point F in relevance to the point E, it holds that $(A, B) \cong (D, F)$ (by th_7_13).
2. From the facts that the point A is symmetry of the point D in relevance to the point E, and the point C is symmetry of the point G in relevance to the point E, it holds that $(A, C) \cong (D, G)$ (by th_7_13).
3. From the facts that the point B is symmetry of the point F in relevance to the point E, and the point C is symmetry of the point G in relevance to the point E, it holds that $(B, C) \cong (F, G)$ (by th_7_13).
4. From the facts that $(A, B) \cong (D, F)$, $(A, C) \cong (D, G)$, and $(B, C) \cong (F, G)$, it holds that $(A, B, C) \cong (D, F, G)$ (by axiom ax_4_4_2).
5. From the facts that bet(A,B,C), and $(A, B, C) \cong (D, F, G)$, it holds that bet($D$,$F$,$G$) (by th_4_6).

Theorem 92 (7.15.2) *Assuming that bet(A,B,C), the point D is symmetry of the point A in relevance to the point E, the point F is symmetry of the point B in relevance to the point E, and the point G is symmetry of the point C in relevance to the point E, show that bet(D,F,G).*

Proof:

1. From the facts that the point D is symmetry of the point A in relevance to the point E, and the point F is symmetry of the point B in relevance to the point E, it holds that $(D,\, F) \cong (A,\, B)$ (by th_7_13).
2. From the facts that $(D,\, F) \cong (A,\, B)$, it holds that $(A,\, B) \cong (D,\, F)$ (by th_2_2).
3. From the facts that the point D is symmetry of the point A in relevance to the point E, and the point G is symmetry of the point C in relevance to the point E, it holds that $(D,\, G) \cong (A,\, C)$ (by th_7_13).
4. From the facts that $(D,\, G) \cong (A,\, C)$, it holds that $(A,\, C) \cong (D,\, G)$ (by th_2_2).
5. From the facts that the point F is symmetry of the point B in relevance to the point E, and the point G is symmetry of the point C in relevance to the point E, it holds that $(F,\, G) \cong (B,\, C)$ (by th_7_13).
6. From the facts that $(F,\, G) \cong (B,\, C)$, it holds that $(B,\, C) \cong (F,\, G)$ (by th_2_2).
7. From the facts that $(A,\, B) \cong (D,\, F)$, $(A,\, C) \cong (D,\, G)$, and $(B,\, C) \cong (F,\, G)$, it holds that $(A,\, B,\, C) \cong (D,\, F,\, G)$ (by axiom ax_4_4_2).
8. From the facts that bet(A,B,C), and $(A,\, B,\, C) \cong (D,\, F,\, G)$, it holds that bet($D$,$F$,$G$) (by th_4_6).

Theorem 93 (7.16.1) *Assuming that $(A,\, B) \cong (C,\, D)$, the point A is symmetry of the point E in relevance to the point F, the point B is symmetry of the point G in relevance to the point F, the point C is symmetry of the point H in relevance to the point F, and the point D is symmetry of the point I in relevance to the point F, show that $(E,\, G) \cong (H,\, I)$.*

Proof:

1. From the facts that the point A is symmetry of the point E in relevance to the point F, and the point B is symmetry of the point G in relevance to the point F, it holds that $(A,\, B) \cong (E,\, G)$ (by th_7_13).
2. From the facts that the point C is symmetry of the point H in relevance to the point F, and the point D is symmetry of the point I in relevance to the point F, it holds that $(C,\, D) \cong (H,\, I)$ (by th_7_13).
3. From the facts that $(A,\, B) \cong (C,\, D)$, and $(A,\, B) \cong (E,\, G)$, it holds that $(C,\, D) \cong (E,\, G)$ (by axiom ax_2).
4. From the facts that $(C,\, D) \cong (E,\, G)$, and $(C,\, D) \cong (H,\, I)$, it holds that $(E,\, G) \cong (H,\, I)$ (by axiom ax_2).

Theorem 94 (7.16.2) *Assuming that $(A,\, B) \cong (C,\, D)$, the point E is symmetry of the point A in relevance to the point F, the point G is symmetry of the point B in relevance to the point F, the point H is symmetry of the point C in relevance to the point F, and the point I is symmetry of the point D in relevance to the point F, show that $(E,\, G) \cong (H,\, I)$.*

Proof:

1. From the facts that the point E is symmetry of the point A in relevance to the point F, it holds that the point A is symmetry of the point E in relevance to the point F (by th_7_7).
2. From the facts that the point G is symmetry of the point B in relevance to the point F, it holds that the point B is symmetry of the point G in relevance to the point F (by th_7_7).
3. From the facts that the point H is symmetry of the point C in relevance to the point F, it holds that the point C is symmetry of the point H in relevance to the point F (by th_7_7).
4. From the facts that the point I is symmetry of the point D in relevance to the point F, it holds that the point D is symmetry of the point I in relevance to the point F (by th_7_7).

5. From the facts that $(A,\ B) \cong (C,\ D)$, the point A is symmetry of the point E in relevance to the point F, the point B is symmetry of the point G in relevance to the point F, the point C is symmetry of the point H in relevance to the point F, and the point D is symmetry of the point I in relevance to the point F, it holds that $(E,\ G) \cong (H,\ I)$ (by th_7_16_1).

Theorem 95 (7.17) *Assuming that the point A is midpoint of points B and C, and the point D is midpoint of points B and C, show that $A = D$.*

Proof:

1. It holds that bet(A,B,B) (by th_3_1).
2. It holds that bet(D,A,A) (by th_3_1).
3. From the facts that the point A is midpoint of points B and C, it holds that bet(B,A,C) and $(A,\ B) \cong (A,\ C)$ (by axiom ax_7_1).
4. From the facts that the point D is midpoint of points B and C, it holds that bet(B,D,C) and $(D,\ B) \cong (D,\ C)$ (by axiom ax_7_1).
5. From the facts that bet(B,A,C), it holds that col(C,B,A) (by axiom ax_4_10_3).
6. From the facts that bet(B,D,C), it holds that col(C,B,D) (by axiom ax_4_10_3).
7. From the facts that bet(D,A,A), it holds that col(A,D,A) (by axiom ax_4_10_3).
8. It holds that $A = B$ or $A \neq B$ (by axiom ax_g1).
    9. Assume that $A = B$.
    10. It holds that $A = C$ or $A \neq C$ (by axiom ax_g1).
        11. Assume that $A = C$.
            12. From the facts that bet(B,D,C), $A = B$ and $A = C$ it holds that bet(A,D,A).
            13. From the facts that bet(A,D,A), and bet(D,A,A), it holds that $A = D$ (by th_3_4).
            This proves the conjecture.
        14. Assume that $A \neq C$.
            15. From the facts that col(C,B,A) and $A = B$ it holds that col(C,A,A).
            16. From the facts that col(C,B,D) and $A = B$ it holds that col(C,A,D).
            Let us prove that $A = D$ by reductio ad absurdum.
                17. Assume that $A \neq D$.
                    18. From the facts that $(A,\ B) \cong (A,\ C)$ and $A = B$ it holds that $(A,\ A) \cong (A,\ C)$.
                    19. From the facts that $(D,\ B) \cong (D,\ C)$ and $A = B$ it holds that $(D,\ A) \cong (D,\ C)$.
                    20. From the facts that $A \neq D$, col(A,D,A), $(A,\ A) \cong (A,\ C)$, and $(D,\ A) \cong (D,\ C)$, it holds that $A = C$ (by th_4_18).
                    21. From the facts that $A = C$, and $A \neq C$ we get a contradiction.
                    Contradiction.
            Therefore, it holds that $A = D$.
            This proves the conjecture.
        This proves the conjecture.
22. Assume that $A \neq B$.
    23. From the facts that $A \neq B$, it holds that $A \in$ line(A,B), $B \in$ line(A,B) and line(A,B)=line(B,A) (by th_6_17).
    Let us prove that $A = D$ by reductio ad absurdum.
        24. Assume that $A \neq D$.
            25. It holds that $B = C$ or $B \neq C$ (by axiom ax_g1).

26. Assume that $B = C$.
    27. From the facts that bet(B,A,C) and $B = C$ it holds that bet(B,A,B).
    28. From the facts that bet(A,B,B), and bet(B,A,B), it holds that $A = B$ (by th_3_4).
    29. From the facts that $A = B$, and $A \neq B$ we get a contradiction.
    Contradiction.
30. Assume that $B \neq C$.
    31. From the fact that $B \neq C$, it holds that $C \neq B$ (by the equality axioms).
    32. From the facts that $C \neq B$, and col(C,B,A), it holds that $A \in$ line(C,B) (by axiom ax_6_14_2).
    33. From the fact that $B \neq C$, it holds that $C \neq B$ (by the equality axioms).
    34. From the facts that $C \neq B$, and col(C,B,D), it holds that $D \in$ line(C,B) (by axiom ax_6_14_2).
    35. From the fact that $B \neq C$, it holds that $C \neq B$ (by the equality axioms).
    36. From the facts that $C \neq B$, it holds that $C \in$ line(C,B), $B \in$ line(C,B) and line(C,B)=line(B,C) (by th_6_17).
    37. From the fact that $B \neq C$, it holds that $C \neq B$ (by the equality axioms).
    38. From the facts that $C \neq B$, $A \in$ line(C,B), $B \in$ line(C,B), and $D \in$ line(C,B), it holds that col(A,B,D) (by th_6_23_2).
    39. From the facts that $A \neq B$, and col(A,B,D), it holds that $D \in$ line(A,B) (by axiom ax_6_14_2).
    40. From the facts that $A \neq B$, $A \in$ line(A,B), $D \in$ line(A,B), and $B \in$ line(A,B), it holds that col(A,D,B) (by th_6_23_2).
    41. From the facts that $A \neq D$, col(A,D,B), (A, B) $\cong$ (A, C), and (D, B) $\cong$ (D, C), it holds that $B = C$ (by th_4_18).
    42. From the facts that $B = C$, and $B \neq C$ we get a contradiction.
    Contradiction.

Therefore, it holds that $A = D$.

This proves the conjecture.

This proves the conjecture.

Theorem 96 (7.18) *Assuming that the point A is symmetry of the point B in relevance to the point C, and the point A is symmetry of the point B in relevance to the point D, show that $C = D$.*

Proof:

1. From the facts that the point A is symmetry of the point B in relevance to the point C, it holds that the point C is midpoint of points A and B (by axiom ax_7_5_1).
2. From the facts that the point A is symmetry of the point B in relevance to the point D, it holds that the point D is midpoint of points A and B (by axiom ax_7_5_1).
3. From the facts that the point C is midpoint of points A and B, and the point D is midpoint of points A and B, it holds that $C = D$ (by th_7_17).

Theorem 97 (7.19.1) *Asuming that the point C is symmetry of the point D in relevance to the point B, the point D is symmetry of the point F in relevance to the point A, the point C is symmetry of the point E in relevance to the point A, and the point E is symmetry of the point F in relevance to the point B, it holds that $A = B$.*

Theorem 98 (7.19.2) *Assuming that the point A is symmetry of the point B in relevance to the point$C$, the point B is symmetry of the point D in relevance to the point$E$, the point A is symmetry of the point F in relevance to the point$E$, the point F is symmetry of the point G in relevance to the point$C$, and $E = C$, show that $D = G$.*

Proof:

1. From the facts that the point A is symmetry of the point B in relevance to the point$C$ and $E = C$ it holds that the point A is symmetry of the point B in relevance to the point$E$.
2. From the facts that the point A is symmetry of the point B in relevance to the point$E$ and $E = C$ it holds that the point A is symmetry of the point B in relevance to the point$C$.
3. From the facts that the point A is symmetry of the point F in relevance to the point$E$ and $E = C$ it holds that the point A is symmetry of the point F in relevance to the point$C$.
4. From the facts that the point B is symmetry of the point D in relevance to the point$E$ and $E = C$ it holds that the point B is symmetry of the point D in relevance to the point$C$.
5. From the facts that the point B is symmetry of the point D in relevance to the point$C$, it holds that the point D is symmetry of the point B in relevance to the point$C$ (by th_7_7).
6. From the facts that the point F is symmetry of the point G in relevance to the point$C$ and $E = C$ it holds that the point F is symmetry of the point G in relevance to the point$E$.
7. From the facts that the point F is symmetry of the point G in relevance to the point$E$ and $E = C$ it holds that the point F is symmetry of the point G in relevance to the point$C$.
8. From the facts that the point F is symmetry of the point G in relevance to the point$C$, it holds that the point G is symmetry of the point F in relevance to the point$C$ (by th_7_7).
9. From the facts that the point A is symmetry of the point B in relevance to the point$C$ and $E = C$ it holds that the point A is symmetry of the point B in relevance to the point$E$.
10. From the facts that the point A is symmetry of the point B in relevance to the point$E$ and $E = C$ it holds that the point A is symmetry of the point B in relevance to the point$C$.
11. From the facts that the point D is symmetry of the point B in relevance to the point$C$ and $E = C$ it holds that the point D is symmetry of the point B in relevance to the point$E$.
12. From the facts that the point D is symmetry of the point B in relevance to the point$E$ and $E = C$ it holds that the point D is symmetry of the point B in relevance to the point$C$.
13. From the facts that the point A is symmetry of the point B in relevance to the point$C$, and the point D is symmetry of the point B in relevance to the point$C$, it holds that $A = D$ (by th_7_8_2).
14. From the facts that the point A is symmetry of the point F in relevance to the point$E$ and $E = C$ it holds that the point A is symmetry of the point F in relevance to the point$C$.
15. From the facts that the point G is symmetry of the point F in relevance to the point$C$ and $E = C$ it holds that the point G is symmetry of the point F in relevance to the point$E$.
16. From the facts that the point G is symmetry of the point F in relevance to the point$E$ and $E = C$ it holds that the point G is symmetry of the point F in relevance to the point$C$.
17. From the facts that the point A is symmetry of the point F in relevance to the point$C$, and the point G is symmetry of the point F in relevance to the point$C$, it holds that $A = G$ (by th_7_8_2).
18. From the facts that $A = D$ and $A = D$ it holds that $A = A$.
19. From the facts that $A = A$, $A = D$ and $A = G$ it holds that $D = G$.

Theorem 99 (7.20) *Asuming that col(A,B,C), and (B, A) $\cong$ (B, C), it holds that $A = C$or the point B is midpoint of points A and C.*

Theorem 100 (7.21) *Asuming that* $\neg$ *col(A,B,C),* $B \neq D$*,* $(A, B) \cong (C, D)$*,* $(B, C) \cong (D, A)$*, col(A,E,C), and col(B,E,D), it holds that the point E is midpoint of points A and C and the point E is midpoint of points B and D.*

Theorem 101 (7.22) *Asuming that bet(A,E,B), bet(C,E,D),* $(E, A) \cong (E, C)$*,* $(E, B) \cong (E, D)$*, the point F is midpoint of points A and C, and the point G is midpoint of points B and D, it holds that bet(F,E,G).*

Theorem 102 (7.25) *Asuming that* $(C, A) \cong (C, B)$*, it holds that there exist a point D such that the point D is midpoint of points A and B.*

Chapter 8

Rechte Winkel (Right Angles)

Definition 40 (8.1.1) *Asuming that points (A,B,C) form a right angle , it holds that there exist a point D such that $(A,\ C) \cong (A,\ D)$ and the point B is midpoint of points C and D.*

Definition 41 (8.1.2) *Asuming that $(A,\ C) \cong (A,\ D)$, and the point B is midpoint of points C and D, it holds that points (A,B,C) form a right angle .*

Theorem 103 (8.2) *Asuming that points (A,B,C) form a right angle , it holds that points (C,B,A) form a right angle .*

Theorem 104 (8.3) *Asuming that points (A,B,C) form a right angle , $A \neq B$, and col(B,A,D), it holds that points (D,B,C) form a right angle .*

Theorem 105 (8.4) *Assuming that points (A,B,C) form a right angle , and the point B is midpoint of points C and D, show that points (A,B,D) form a right angle .*

Proof:

1. From the facts that the point B is midpoint of points C and D, it holds that the point B is midpoint of points D and C (by th_7_2).
2. From the facts that points (A,B,C) form a right angle , there exist a point E such that $(A,\ C) \cong (A,\ E)$ and the point B is midpoint of points C and E (by axiom ax_8_1_1).
3. From the facts that the point B is midpoint of points C and D, and the point B is midpoint of points C and E, it holds that $D = E$ (by th_7_4_2).
4. From the facts that $(A,\ C) \cong (A,\ E)$ and $D = E$ it holds that $(A,\ C) \cong (A,\ D)$.
5. From the facts that $(A,\ C) \cong (A,\ D)$, it holds that $(A,\ D) \cong (A,\ C)$ (by th_2_2).
6. From the facts that $(A,\ D) \cong (A,\ C)$, and the point B is midpoint of points D and C, it holds that points (A,B,D) form a right angle (by axiom ax_8_1_2).

Theorem 106 (8.5) *Show that points (A,B,B) form a right angle .*

Proof:

1. It holds that the point B is midpoint of points B and B (by th_7_3_2).
2. It holds that $(A,\ B) \cong (A,\ B)$ (by th_2_1).
3. From the facts that $(A,\ B) \cong (A,\ B)$, and the point B is midpoint of points B and B, it holds that points (A,B,B) form a right angle (by axiom ax_8_1_2).

Theorem 107 (8.6) *Asuming that points (A,B,C) form a right angle , points (D,B,C) form a right angle , and bet(A,C,D), it holds that B = C.*

Theorem 108 (8.7) *Asuming that points (A,B,C) form a right angle , and points (A,C,B) form a right angle , it holds that B = C.*

Theorem 109 (8.8) *Assuming that points (A,B,A) form a right angle , show that A = B.*

Proof:

1. It holds that bet(A,A,A) (by th_3_1).
2. From the facts that points (A,B,A) form a right angle , points (A,B,A) form a right angle , and bet(A,A,A), it holds that $B = A$ (by th_8_6).

Theorem 110 (8.9) *Assuming that points (A,B,C) form a right angle , and col(A,B,C), show that A = B or C = B.*

Proof:

1. From the facts that col(A,B,C), it holds that col(B,C,A), col(C,A,B), col(C,B,A), col(B,A,C) and col(A,C,B) (by th_4_11).
2. It holds that $A = B$ or $A \neq B$ (by axiom ax_g1).
   3. Assume that $A = B$.
      This proves the conjecture.
   4. Assume that $A \neq B$.
      5. From the facts that points (A,B,C) form a right angle , $A \neq B$, and col(B,A,C), it holds that points (C,B,C) form a right angle (by th_8_3).
      6. From the facts that points (C,B,C) form a right angle , it holds that $C = B$ (by th_8_8).
         This proves the conjecture.

Theorem 111 (8.10) *Asuming that points (A,B,C) form a right angle , and (A, B, C) $\cong$ (D, E, F), it holds that points (D,E,F) form a right angle .*

Definition 42 (8.11.1.1) *Asuming that the point A is intersection point of ortogonal lines (B,C) and (D,E), B $\neq$ A, and D $\neq$ A, it holds that B $\neq$ C, D $\neq$ E, A $\in$ line(B,C), A $\in$ line(D,E) and points (B,A,D) form a right angle .*

Definition 43 (8.11.1.2) *Asuming that the point A is intersection point of ortogonal lines (B,C) and (D,E), B $\neq$ A, and E $\neq$ A, it holds that B $\neq$ C, D $\neq$ E, A $\in$ line(B,C), A $\in$ line(D,E) and points (B,A,E) form a right angle .*

Definition 44 (8.11.1.3) *Asuming that the point A is intersection point of ortogonal lines (B,C) and (D,E), C $\neq$ A, and D $\neq$ A, it holds that B $\neq$ C, D $\neq$ E, A $\in$ line(B,C), A $\in$ line(D,E) and points (C,A,D) form a right angle .*

Definition 45 (8.11.1.4) *Asuming that the point A is intersection point of ortogonal lines (B,C) and (D,E), C $\neq$ A, and E $\neq$ A, it holds that B $\neq$ C, D $\neq$ E, A $\in$ line(B,C), A $\in$ line(D,E) and points (C,A,E) form a right angle .*

Definition 46 (8.11.1.5) *Asuming that B $\neq$ A, D $\neq$ A, B $\neq$ C, D $\neq$ E, A $\in$ line(B,C), A $\in$ line(D,E), and points (B,A,D) form a right angle , it holds that the point A is intersection point of ortogonal lines (B,C) and (D,E).*

Definition 47 (8.11.1.6) *Asuming that B $\neq$ A, E $\neq$ A, B $\neq$ C, D $\neq$ E, A $\in$ line(B,C), A $\in$ line(D,E), and points (B,A,E) form a right angle , it holds that the point A is intersection point of ortogonal lines (B,C) and (D,E).*

Definition 48 (8.11.1.7) *Asuming that C $\neq$ A, D $\neq$ A, B $\neq$ C, D $\neq$ E, A $\in$ line(B,C), A $\in$ line(D,E), and points (C,A,D) form a right angle , it holds that the point A is intersection point of ortogonal lines (B,C) and (D,E).*

Definition 49 (8.11.1.8) *Asuming that C $\neq$ A, E $\neq$ A, B $\neq$ C, D $\neq$ E, A $\in$ line(B,C), A $\in$ line(D,E), and points (C,A,E) form a right angle , it holds that the point A is intersection point of ortogonal lines (B,C) and (D,E).*

Definition 50 (8.11.2.1) *Asuming that lines (A,B) and (C,D) are ortogonal, it holds that there exist a point E such that the point E is intersection point of ortogonal lines (A,B) and (C,D).*

Definition 51 (8.11.2.2) *Asuming that the point A is intersection point of ortogonal lines (B,C) and (D,E), it holds that lines (B,C) and (D,E) are ortogonal.*

Theorem 112 (8.12) *Asuming that the point A is intersection point of ortogonal lines (B,C) and (D,E), it holds that the point A is intersection point of ortogonal lines (D,E) and (B,C).*

Theorem 113 (8.13.1) *Asuming that the point A is intersection point of ortogonal lines (B,C) and (D,E), it holds that there exist a point F and a point G such that* $B \neq C$*,* $D \neq E$*,* $A \in$ *line(B,C),* $A \in$ *line(D,E),* $F \in$ *line(B,C),* $G \in$ *line(D,E),* $F \neq A$*,* $G \neq A$ *and points (F,A,G) form a right angle .*

Theorem 114 (8.13.2) *Asuming that* $B \neq C$*,* $D \neq E$*,* $A \in$ *line(B,C),* $A \in$ *line(D,E),* $F \in$ *line(B,C), G* $\in$ *line(D,E),* $F \neq A$*,* $G \neq A$*, and points (F,A,G) form a right angle , it holds that the point A is intersection point of ortogonal lines (B,C) and (D,E).*

Theorem 115 (8.14.1) *Asuming that lines (A,B) and (C,D) are ortogonal, it holds that* $A \neq B$*,* $C \neq D$ *and line(A,B)≠line(C,D).*

Theorem 116 (8.14.2.1) *Asuming that the point A is intersection point of ortogonal lines (B,C) and (D,E), it holds that lines (B,C) and (D,E) are ortogonal and point A is the intersection of line(B,C) and line(D,E).*

Theorem 117 (8.14.2.2) *Asuming that lines (B,C) and (D,E) are ortogonal, and point A is the intersection of line(B,C) and line(D,E), it holds that the point A is intersection point of ortogonal lines (B,C) and (D,E).*

Theorem 118 (8.14.3) *Asuming that the point E is intersection point of ortogonal lines (A,B) and (C,D), and the point F is intersection point of ortogonal lines (A,B) and (C,D), it holds that* $E = F$*.*

Theorem 119 (8.15.1) *Assuming that* $A \neq B$*, col(A,B,C), and lines (A,B) and (D,C) are ortogonal, show that the point C is intersection point of ortogonal lines (A,B) and (D,C).*

Proof:

1. From the facts that lines (A,B) and (D,C) are ortogonal, it holds that $A \neq B$, $D \neq C$ and line(A,B)≠line(D,C) (by th_8_14_1).
2. From the facts that $D \neq C$, it holds that $D \in$ line(D,C), $C \in$ line(D,C) and line(D,C)=line(C,D) (by th_6_17).
3. From the facts that $A \neq B$, and col(A,B,C), it holds that $C \in$ line(A,B) (by axiom ax_6_14_2).
4. From the facts that $A \neq B$, $D \neq C$, $C \in$ line(A,B), $C \in$ line(D,C), and line(A,B)≠line(D,C), it holds that point C is the intersection of line(A,B) and line(D,C) (by axiom ax_6_22_2).
5. From the facts that lines (A,B) and (D,C) are ortogonal, and point C is the intersection of line(A,B) and line(D,C), it holds that the point C is intersection point of ortogonal lines (A,B) and (D,C) (by th_8_14_2_2).

Theorem 120 (8.15.2) *Assuming that $A \neq B$, col(A,B,C), and the point C is intersection point of ortogonal lines (A,B) and (D,C), show that lines (A,B) and (D,C) are ortogonal.*

Proof:

1. From the facts that the point C is intersection point of ortogonal lines (A,B) and (D,C), it holds that lines (A,B) and (D,C) are ortogonal (by axiom ax_8_11_2_2).

Theorem 121 (8.16.1) *Asuming that $A \neq B$, col(A,B,D), col(A,B,E), $E \neq D$, and lines (A,B) and (C,D) are ortogonal, it holds that ¬ col(A,B,C) and points (C,D,E) form a right angle .*

Theorem 122 (8.16.2) *Asuming that $A \neq B$, col(A,B,D), col(A,B,E), $E \neq D$, ¬ col(A,B,C), and points (C,D,E) form a right angle , it holds that lines (A,B) and (C,D) are ortogonal.*

Theorem 123 (8.20.1) *Asuming that points (A,B,C) form a right angle , the point C is symmetry of the point E in relevance to the point A, the point C is symmetry of the point F in relevance to the point B, and the point D is midpoint of points E and F, it holds that points (B,A,D) form a right angle .*

Theorem 124 (8.18.1) *Asuming that ¬ col(A,B,C), it holds that there exist a point D such that col(A,B,D) and lines (A,B) and (C,D) are ortogonal.*

Theorem 125 (8.18.2) *Assuming that ¬ col(A,B,C), col(A,B,D), lines (A,B) and (C,D) are ortogonal, col(A,B,E), and lines (A,B) and (C,E) are ortogonal, show that $D = E$.*

Proof:

1. From the facts that lines (A,B) and (C,D) are ortogonal, it holds that $A \neq B$, $C \neq D$ and line(A,B)≠line(C,D) (by th_8_14_1).
2. From the facts that lines (A,B) and (C,E) are ortogonal, it holds that $A \neq B$, $C \neq E$ and line(A,B)≠line(C,E) (by th_8_14_1).
3. It holds that $A = D$ or $A \neq D$ (by axiom ax_g1).
   4. Assume that $A = D$.
      Let us prove that $A = E$ by reductio ad absurdum.
      5. Assume that $A \neq E$.
         6. From the facts that col(A,B,D) and $A = D$ it holds that col(A,B,A).
         7. From the fact that $A \neq E$, it holds that $E \neq A$ (by the equality axioms).
         8. From the facts that lines (A,B) and (C,D) are ortogonal and $A = D$ it holds that lines (A,B) and (C,A) are ortogonal.
         9. From the facts that $A \neq B$, col(A,B,A), col(A,B,E), $E \neq A$, and lines (A,B) and (C,A) are ortogonal, it holds that ¬ col(A,B,C) and points (C,A,E) form a right angle (by th_8_16_1).
         10. From the facts that col(A,B,D) and $A = D$ it holds that col(A,B,A).

11. From the facts that $A \neq B$, col(A,B,E), col(A,B,A), $A \neq E$, and lines (A,B) and (C,E) are ortogonal, it holds that ¬ col(A,B,C) and points (C,E,A) form a right angle (by th_8_16_1).

12. From the facts that points (C,A,E) form a right angle , and points (C,E,A) form a right angle , it holds that $A = E$ (by th_8_7).

13. From the facts that $A = E$, and $A \neq E$ we get a contradiction.

    Contradiction.

Therefore, it holds that $A = E$.

This proves the conjecture.

14. Assume that $A \neq D$.

15. It holds that $B = D$ or $B \neq D$ (by axiom ax_g1).

16. Assume that $B = D$.

Let us prove that $B = E$ by reductio ad absurdum.

17. Assume that $B \neq E$.

18. From the facts that col(A,B,D) and $B = D$ it holds that col(A,B,B).

19. From the fact that $B \neq E$, it holds that $E \neq B$ (by the equality axioms).

20. From the facts that lines (A,B) and (C,D) are ortogonal and $B = D$ it holds that lines (A,B) and (C,B) are ortogonal.

21. From the facts that $A \neq B$, col(A,B,B), col(A,B,E), $E \neq B$, and lines (A,B) and (C,B) are ortogonal, it holds that ¬ col(A,B,C) and points (C,B,E) form a right angle (by th_8_16_1).

22. From the facts that col(A,B,D) and $B = D$ it holds that col(A,B,B).

23. From the facts that $A \neq B$, col(A,B,E), col(A,B,B), $B \neq E$, and lines (A,B) and (C,E) are ortogonal, it holds that ¬ col(A,B,C) and points (C,E,B) form a right angle (by th_8_16_1).

24. From the facts that points (C,B,E) form a right angle , and points (C,E,B) form a right angle , it holds that $B = E$ (by th_8_7).

25. From the facts that $B = E$, and $B \neq E$ we get a contradiction.

    Contradiction.

Therefore, it holds that $B = E$.

This proves the conjecture.

26. Assume that $B \neq D$.

Let us prove that $D = E$ by reductio ad absurdum.

27. Assume that $D \neq E$.

28. From the fact that $D \neq E$, it holds that $E \neq D$ (by the equality axioms).

29. From the facts that $A \neq B$, col(A,B,D), col(A,B,E), $E \neq D$, and lines (A,B) and (C,D) are ortogonal, it holds that ¬ col(A,B,C) and points (C,D,E) form a right angle (by th_8_16_1).

30. From the facts that $A \neq B$, col(A,B,E), col(A,B,D), $D \neq E$, and lines (A,B) and (C,E) are ortogonal, it holds that ¬ col(A,B,C) and points (C,E,D) form a right angle (by th_8_16_1).

31. From the facts that points (C,D,E) form a right angle , and points (C,E,D) form a right angle , it holds that $D = E$ (by th_8_7).

32. From the facts that $D = E$, and $D \neq E$ we get a contradiction.

    Contradiction.

Therefore, it holds that $D = E$.

This proves the conjecture.

This proves the conjecture.

Theorem 126 (8.20.2) *Assuming that points (A,B,C) form a right angle , the point C is symmetry of the point D in relevance to the point A, the point C is symmetry of the point E in relevance to the point B, the point F is midpoint of points D and E, and $B \neq C$, show that $A \neq F$.*

Proof:

1. From the facts that the point C is symmetry of the point D in relevance to the point A, it holds that the point A is midpoint of points C and D (by axiom ax_7_5_1).
2. From the facts that the point A is midpoint of points C and D, it holds that the point A is midpoint of points D and C (by th_7_2).
3. From the facts that the point C is symmetry of the point E in relevance to the point B, it holds that the point B is midpoint of points C and E (by axiom ax_7_5_1).
4. It holds that bet(B,C,C) (by th_3_1).
5. From the facts that the point B is midpoint of points C and E, it holds that bet(C,B,E) and $(B, C) \cong (B, E)$ (by axiom ax_7_1).

Let us prove that $A \neq F$ by reductio ad absurdum.

6. Assume that $A = F$.
7. From the facts that the point F is midpoint of points D and E and $A = F$ it holds that the point A is midpoint of points D and E.
8. From the facts that the point A is midpoint of points D and C, and the point A is midpoint of points D and E, it holds that $C = E$ (by th_7_4_2).
9. From the facts that bet(C,B,E) and $C = E$ it holds that bet(C,B,C).
10. From the facts that bet(B,C,C), and bet(C,B,C), it holds that $B = C$ (by th_3_4).
11. From the facts that $B = C$, and $B \neq C$ we get a contradiction.

Contradiction.

Therefore, it holds that $A \neq F$.

This proves the conjecture.

Theorem 127 (8.21) *Asuming that $A \neq B$, it holds that there exist a point D and a point E such that lines (A,B) and (D,A) are ortogonal, col(A,B,E) and bet(C,E,D).*

Theorem 128 (8.22.1) *There exist a point C such that the point C is midpoint of points A and B.*

Theorem 129 (8.22.2) *Assuming that the point A is midpoint of points B and C, and the point D is midpoint of points B and C, show that $A = D$.*

Proof:

1. From the facts that the point A is midpoint of points B and C, and the point D is midpoint of points B and C, it holds that $A = D$ (by th_7_17).

Theorem 130 (8.24) *Asuming that lines (C,A) and (A,B) are ortogonal, lines (D,B) and (A,B) are ortogonal, col(A,B,E), bet(C,E,D), bet(B,F,D), and $(A, C) \cong (B, F)$, it holds that there exist a point G such that the point G is midpoint of points A and B and the point G is midpoint of points C and F.*

Chapter 9

Halbebenen und Ebenen, Unterräume (Half-planes and Planes, Subspaces)

Definition 52 (9.1.1) *Asuming that points C and D are on different sides of line (A,B), it holds that there exist a point E such that $A \neq B$, $C \notin$ line(A,B), $D \notin$ line(A,B), $E \in$ line(A,B) and bet(C,E,D).*

Definition 53 (9.1.2) *Asuming that $A \neq B$, $C \notin$ line(A,B), $D \notin$ line(A,B), $E \in$ line(A,B), and bet(C,E,D), it holds that points C and D are on different sides of line (A,B).*

Theorem 131 (9.2) *Assuming that points A and B are on different sides of line (C,D), show that points B and A are on different sides of line (C,D).*

Proof:

1. From the facts that points A and B are on different sides of line (C,D), there exist a point E such that $C \neq D$, $A \notin$ line(C,D), $B \notin$ line(C,D), $E \in$ line(C,D) and bet(A,E,B) (by axiom ax_9_1_1).
2. From the facts that bet(A,E,B), it holds that bet(B,E,A) (by th_3_2).
3. From the facts that $C \neq D$, $B \notin$ line(C,D), $A \notin$ line(C,D), $E \in$ line(C,D), and bet(B,E,A), it holds that points B and A are on different sides of line (C,D) (by axiom ax_9_1_2).

Theorem 132 (9.3) *Asuming that points C and E are on different sides of line (A,B), $A \neq B$, $F \in$ line(A,B), the point F is midpoint of points C and E, $G \in$ line(A,B), and out(G,C,D), it holds that points D and E are on different sides of line (A,B).*

Theorem 133 (9.4.1) *Asuming that points C and D are on different sides of line (A,B), $A \neq B$, $E \in$ line(A,B), lines (A,B) and (C,E) are ortogonal, $F \in$ line(A,B), lines (A,B) and (D,F) are ortogonal, the point G is midpoint of points E and F, out(E,H,C), and the point H is symmetry of the point I in relevance to the point G, it holds that out(F,I,D).*

Theorem 134 (9.4.2) *Asuming that points C and D are on different sides of line (A,B), $A \neq B$, $E \in$ line(A,B), lines (A,B) and (C,E) are ortogonal, $F \in$ line(A,B), lines (A,B) and (D,F) are ortogonal, the point G is midpoint of points E and F, out(F,I,D), and the point H is symmetry of the point I in relevance to the point G, it holds that out(E,H,C).*

Theorem 135 (9.4.3) *Asuming that points C and D are on different sides of line (A,B), $A \neq B$, $E \in$ line(A,B), lines (A,B) and (C,E) are ortogonal, $F \in$ line(A,B), lines (A,B) and (D,F) are ortogonal, out(E,G,C), and out(F,H,D), it holds that points G and H are on different sides of line (A,B).*

Theorem 136 (9.5) *Asuming that $A \neq B$, points C and E are on different sides of line (A,B), $F \in$ line(A,B), and out(F,C,D), it holds that points D and E are on different sides of line (A,B).*

Theorem 137 (9.6) *Asuming that bet(A,C,D), and bet(B,E,C), it holds that there exist a point F such that bet(A,F,B) and bet(D,E,F).*

Definition 54 (9.7.1) *Asuming that points C and D are on the same side of line (A,B), it holds that there exist a point E such that $A \neq B$, points C and E are on different sides of line (A,B) and points D and E are on different sides of line (A,B).*

Definition 55 (9.7.2) *Asuming that $A \neq B$, points C and E are on different sides of line (A,B), and points D and E are on different sides of line (A,B), it holds that points C and D are on the same side of line (A,B).*

Theorem 138 (9.8.1) *Assuming that points A and B are on different sides of line (C,D), and points E and B are on different sides of line (C,D), show that points A and E are on the same side of line (C,D).*

Proof:

Let us prove that $C \neq D$ by reductio ad absurdum.

1. Assume that $C = D$.

2. From the facts that points A and B are on different sides of line (C,D) and $C = D$ it holds that points A and B are on different sides of line (C,C).

3. From the facts that points A and B are on different sides of line (C,C), there exist a point F such that $C \neq C$, $A \notin$ line(C,C), $B \notin$ line(C,C), $F \in$ line(C,C) and bet(A,F,B) (by axiom ax_9_1_1).

4. From the facts that $C \neq C$, and $C = C$ we get a contradiction.

Contradiction.

Therefore, it holds that $C \neq D$.

5. From the facts that $C \neq D$, points A and B are on different sides of line (C,D), and points E and B are on different sides of line (C,D), it holds that points A and E are on the same side of line (C,D) (by axiom ax_9_7_2).

This proves the conjecture.

Theorem 139 (9.8.2) *Asuming that points C and E are on different sides of line (A,B), and points C and D are on the same side of line (A,B), it holds that points D and E are on different sides of line (A,B).*

Theorem 140 (9.9) *Asuming that points C and D are on different sides of line (A,B), it holds that points C and D are not on the same side of line (A,B).*

Theorem 141 (9.10) *Asuming that $A \neq B$, and $C \notin$ line(A,B), it holds that there exist a point D such that points C and D are on different sides of line (A,B).*

Theorem 142 (9.11) *Assuming that $A \neq B$, and $C \notin$ line(A,B), show that points C and C are on the same side of line (A,B).*

Proof:

1. From the facts that $A \neq B$, and $C \notin$ line(A,B), there exist a point D such that points C and D are on different sides of line (A,B) (by th_9_10).
2. From the facts that points C and D are on different sides of line (A,B), and points C and D are on different sides of line (A,B), it holds that points C and C are on the same side of line (A,B) (by th_9_8_1).

Theorem 143 (9.12) *Assuming that points A and B are on the same side of line (C,D), show that points B and A are on the same side of line (C,D).*

Proof:

1. It holds that $A = B$ or $A \neq B$ (by axiom ax_g1).
   2. Assume that $A = B$.
      3. From the facts that points A and B are on the same side of line (C,D) and $A = B$ it holds that points A and A are on the same side of line (C,D).
      4. From the facts that points A and A are on the same side of line (C,D) and $A = B$ it holds that points B and A are on the same side of line (C,D).
         This proves the conjecture.
   5. Assume that $A \neq B$.
      6. From the facts that points A and B are on the same side of line (C,D), there exist a point E such that $C \neq D$, points A and E are on different sides of line (C,D) and points B and E are on different sides of line (C,D) (by axiom ax_9_7_1).
      7. From the facts that points B and E are on different sides of line (C,D), and points A and E are on different sides of line (C,D), it holds that points B and A are on the same side of line (C,D) (by th_9_8_1).
         This proves the conjecture.

Theorem 144 (9.13) *Assuming that points A and B are on the same side of line (C,D), and points B and E are on the same side of line (C,D), show that points A and E are on the same side of line (C,D).*

Proof:

1. It holds that $A = B$ or $A \neq B$ (by axiom ax_g1).
2. Assume that $A = B$.
3. From the facts that points B and E are on the same side of line (C,D) and $A = B$ it holds that points A and E are on the same side of line (C,D).

This proves the conjecture.

4. Assume that $A \neq B$.

Let us prove that $C \neq D$ by reductio ad absurdum.

5. Assume that $C = D$.
6. From the facts that points A and B are on the same side of line (C,D) and $C = D$ it holds that points A and B are on the same side of line (C,C).
7. From the facts that points A and B are on the same side of line (C,C), there exist a point F such that $C \neq C$, points A and F are on different sides of line (C,C) and points B and F are on different sides of line (C,C) (by axiom ax_9_7_1).
8. From the facts that $C \neq C$, and $C = C$ we get a contradiction.

Contradiction.

Therefore, it holds that $C \neq D$.

9. From the facts that points A and B are on the same side of line (C,D), there exist a point F such that $C \neq D$, points A and F are on different sides of line (C,D) and points B and F are on different sides of line (C,D) (by axiom ax_9_7_1).
10. From the facts that points B and F are on different sides of line (C,D), and points B and E are on the same side of line (C,D), it holds that points E and F are on different sides of line (C,D) (by th_9_8_2).
11. From the facts that points A and F are on different sides of line (C,D), and points E and F are on different sides of line (C,D), it holds that points A and E are on the same side of line (C,D) (by th_9_8_1).

This proves the conjecture.

Theorem 145 (9.17) *Asuming that points C and D are on the same side of line (A,B), and bet(C,E,D), it holds that points E and C are on the same side of line (A,B).*

Theorem 146 (9.18.1) *Asuming that $A \neq B$, $C \in$ line(A,B), col(D,E,C), and points D and E are on different sides of line (A,B), it holds that bet(D,C,E), $D \notin$ line(A,B) and $E \notin$ line(A,B).*

Theorem 147 (9.18.2) *Assuming that $A \neq B$, $C \in$ line(A,B), col(D,E,C), bet(D,C,E), $D \notin$ line(A,B), and $E \notin$ line(A,B), show that points D and E are on different sides of line (A,B).*

Proof:

1. From the facts that $A \neq B$, $D \notin$ line(A,B), $E \notin$ line(A,B), $C \in$ line(A,B), and bet(D,C,E), it holds that points D and E are on different sides of line (A,B) (by axiom ax_9_1_2).

Theorem 148 (9.19.1) *Asuming that $A \neq B$, $C \in line(A,B)$, $col(D,E,C)$, and points D and E are on the same side of line (A,B), it holds that $out(C,D,E)$ and $D \notin line(A,B)$.*

Theorem 149 (9.19.2) *Assuming that $A \neq B$, $C \in line(A,B)$, $col(D,E,C)$, $out(C,D,E)$, and $D \notin line(A,B)$, show that points D and E are on the same side of line (A,B).*

Proof:

1. From the facts that $A \neq B$, and $D \notin \text{line}(A,B)$, there exist a point F such that points D and F are on different sides of line (A,B) (by th_9_10).
2. From the facts that $A \neq B$, points D and F are on different sides of line (A,B), $C \in \text{line}(A,B)$, and $\text{out}(C,D,E)$, it holds that points E and F are on different sides of line (A,B) (by th_9_5).
3. From the facts that points D and F are on different sides of line (A,B), and points E and F are on different sides of line (A,B), it holds that points D and E are on the same side of line (A,B) (by th_9_8_1).

Theorem 150 (9.31) *Asuming that points A and B are on the same side of line (C,D), and points A and D are on the same side of line (C,B), it holds that points D and B are on different sides of line (C,A).*

Chapter 10

Geradenspiegelungen (Line reflexivity)

Theorem 151 (10.2) *Asuming that $A \neq B$, it holds that the point D is an image of the point C in relation to the line (A,B).*

Definition 56 (10.3.1) *Asuming that $C \neq D$, and the point B is an image of the point A in relation to the line (C,D), it holds that there exist a point E such that the point E is midpoint of points A and B, $E \in$ line(C,D) and lines (C,D) and (A,B) are ortogonalor and a point E such that the point E is midpoint of points A and B, $E \in$ line(C,D) and $A = B$.*

Definition 57 (10.3.2.1) *Asuming that $C \neq D$, the point E is midpoint of points A and B, $E \in$ line(C,D), and lines (C,D) and (A,B) are ortogonal, it holds that the point B is an image of the point A in relation to the line (C,D).*

Definition 58 (10.3.2.2) *Asuming that $B \neq C$, the point D is midpoint of points A and A, and $D \in$ line(B,C), it holds that the point A is an image of the point A in relation to the line (B,C).*

Definition 59 (10.3.3) *Asuming that the point B is an image of the point A in relation to the points (C,D), it holds that $C \neq D$ and the point B is an image of the point A in relation to the line (C,D) or $C = D$ and the point C is midpoint of points A and B.*

Definition 60 (10.3.4.1) *Asuming that $C \neq D$, and the point B is an image of the point A in relation to the line (C,D), it holds that the point B is an image of the point A in relation to the points (C,D).*

Definition 61 (10.3.4.2) *Asuming that the point C is midpoint of points A and B, it holds that the point B is an image of the point A in relation to the points (C,C).*

Definition 62 (is.image.spec.in.1) *Asuming that $D \neq E$, and the point A is center of the image of the point B to the point C in relation to the line (D,E), it holds that the point A is midpoint of points C and B, col(D,E,A) and lines (D,E) and (C,B) are ortogonalor the point A is midpoint of points C and B, col(D,E,A) and $C = B$.*

Definition 63 (is.image.spec.in.2) *Asuming that $D \neq E$, the point A is midpoint of points C and B, col(D,E,A), and lines (D,E) and (C,B) are ortogonal, it holds that the point A is center of the image of the point B to the point C in relation to the line (D,E).*

Definition 64 (is.image.spec.in.3) *Asuming that $C \neq D$, the point A is midpoint of points B and B, and col(C,D,A), it holds that the point A is center of the image of the point B to the point B in relation to the line (C,D).*

Definition 65 (is.image.spec.in.gen.1) *Asuming that the point A is center of the image of the point B to the point C in relation to the points (D,E), it holds that $D \neq E$ and the point A is center of the image of the point B to the point C in relation to the line (D,E) or $D = E$, $D = A$ and the point A is midpoint of points C and B.*

Definition 66 (is.image.spec.in.gen.2) *Asuming that $D \neq E$, and the point A is center of the image of the point B to the point C in relation to the line (D,E), it holds that the point A is center of the image of the point B to the point C in relation to the points (D,E).*

Definition 67 (is.image.spec.in.gen.3) *Asuming that the point A is midpoint of points C and B, it holds that there exist a point G such that the point A is center of the image of the point B to the point C in relation to the points (A,A).*

Theorem 152 (10.4) *Asuming that $C \neq D$ and the point B is an image of the point A in relation to the line (C,D), it holds that the point A is an image of the point B in relation to the line (C,D).*

Theorem 153 (10.5) *Asuming that $A \neq B$, the point D is an image of the point C in relation to the line (A,B), and the point E is an image of the point D in relation to the line (A,B), it holds that $C = E$.*

Theorem 154 (10.6.1) *Assuming that $A \neq B$, show that there exist a point D such that the point C is an image of the point D in relation to the points (A,B).*

Proof:

1. From the facts that $A \neq B$, there exist a point F such that the point F is an image of the point C in relation to the points (A,B) (by th_10_2).
2. From the facts that the point F is an image of the point C in relation to the points (A,B), it holds that the point C is an image of the point F in relation to the points (A,B) (by th_10_4).

Theorem 155 (10.6.2) *Assuming that $A \neq B$, the point C is an image of the point D in relation to the points (A,B), and the point C is an image of the point E in relation to the points (A,B), show that $D = E$.*

Proof:

1. From the facts that the point C is an image of the point E in relation to the points (A,B), it holds that the point E is an image of the point C in relation to the points (A,B) (by th_10_4).
2. From the facts that $A \neq B$, the point C is an image of the point D in relation to the points (A,B), and the point E is an image of the point C in relation to the points (A,B), it holds that $D = E$ (by th_10_5).

Theorem 156 (10.7) *Assuming that $A \neq B$, the point C is an image of the point D in relation to the points (A,B), and the point C is an image of the point E in relation to the points (A,B), show that $D = E$.*

Proof:

1. From the facts that $A \neq B$, the point C is an image of the point D in relation to the points (A,B), and the point C is an image of the point E in relation to the points (A,B), it holds that $D = E$ (by th_10_6_2).

Theorem 157 (10.8.1) *Asuming that $A \neq B$, and the point C is an image of the point C in relation to the line (A,B), it holds that $C \in$ line(A,B).*

Theorem 158 (10.8.2) *Assuming that $A \neq B$, and $C \in$ line(A,B), show that the point C is an image of the point C in relation to the points (A,B).*

Proof:

1. It holds that the point C is midpoint of points C and C (by th_7_3_2).
2. From the facts that $A \neq B$, the point C is midpoint of points C and C, and $C \in$ line(A,B), it holds that the point C is an image of the point C in relation to the line (A,B) (by axiom ax_10_3_2_2).
3. From the facts that $A \neq B$, and the point C is an image of the point C in relation to the line (A,B), it holds that the point C is an image of the point C in relation to the points (A,B) (by axiom ax_10_3_4_1).

Theorem 159 (10.10) *Asuming that $A \neq B$, the point E is an image of the point C in relation to the line (A,B), and the point F is an image of the point D in relation to the line (A,B), it holds that $(C, D) \cong (E, F)$.*

Theorem 160 (10.12) *Asuming that points (A,B,C) form a right angle , points (D,E,F) form a right angle , $(A, B) \cong (D, E)$, and $(B, C) \cong (E, F)$, it holds that $(A, C) \cong (D, F)$.*

Theorem 161 (10.14) *Asuming that $A \neq B$, the point D is an image of the point C in relation to the line (A,B), and $C \notin$ line(A,B), it holds that points C and D are on different sides of line (A,B).*

Theorem 162 (10.15) *Asuming that $A \neq B$, $C \in$ line(A,B), and $D \notin$ line(A,B), it holds that there exist a point E such that lines (A,B) and (E,C) are ortogonal and points E and D are on the same side of line (A,B).*

Theorem 163 (10.16.1) *Asuming that $\neg$ col(A,B,C), $\neg$ col(D,E,F), and $(A, B) \cong (D, E)$, it holds that there exist a point G such that $(A, B, C) \cong (D, E, G)$ and points G and F are on the same side of line (D,E).*

Theorem 164 (10.16.2) *Asuming that $\neg$ col(A,B,C), $\neg$ col(D,E,F), $(A, B) \cong (D, E)$, $(A, B, C) \cong (D, E, G)$, points G and F are on the same side of line (D,E), $(A, B, C) \cong (D, E, H)$, and points H and F are on the same side of line (D,E), it holds that $G = H$.*

Chapter 11

Kongruenz und Grössenvergleich von Winkeln, Kongruenzsätze, Orthogonalität für Unterräume (Congruence and Comparison of Angles, Congruence theorem, Orthogonality for Subspaces)

Definition 68 (11.2.1) *Asuming that angle(A,B,C) $\cong$ angle(D,E,F) , it holds that there exist a point G and a point H and a point I and a point J such that A $\neq$ B, C $\neq$ B, D $\neq$ E, F $\neq$ E, bet(B,A,G), (A, G) $\cong$ (E, D), bet(B,C,H), (C, H) $\cong$ (E, F), bet(E,D,I), (D, I) $\cong$ (B, A), bet(E,F,J), (F, J) $\cong$ (B, C) and (G, H) $\cong$ (I, J).*

Definition 69 (11.2.2) *Asuming that A $\neq$ B, C $\neq$ B, D $\neq$ E, F $\neq$ E, bet(B,A,G), (A, G) $\cong$ (E, D), bet(B,C,H), (C, H) $\cong$ (E, F), bet(E,D,I), (D, I) $\cong$ (B, A), bet(E,F,J), (F, J) $\cong$ (B, C), and (G, H) $\cong$ (I, J), it holds that angle(A,B,C) $\cong$ angle(D,E,F) .*

Theorem 165 (11.3.1) *Asuming that angle(A,B,C) $\cong$ angle(D,E,F) , it holds that there exist a point G and a point H and a point I and a point J such that out(B,G,A), out(B,H,C), out(E,I,D), out(E,J,F) and (G, B, H) $\cong$ (I, E, J).*

Theorem 166 (11.3.2) *Asuming that out(B,G,A), out(B,H,C), out(E,I,D), out(E,J,F), and (G, B, H) $\cong$ (I, E, J), it holds that angle(A,B,C) $\cong$ angle(D,E,F) .*

Theorem 167 (11.4.1) *Asuming that angle(A,B,C) $\cong$ angle(D,E,F) , A $\neq$ B, C $\neq$ B, D $\neq$ E, F $\neq$ E, out(B,G,A), out(B,H,C), out(E,I,D), out(E,J,F), (B, G) $\cong$ (E, I), and (B, H) $\cong$ (E, J), it holds that (G, H) $\cong$ (I, J).*

Theorem 168 (11.4.2) *Asuming that $A \neq B$, $C \neq B$, $D \neq E$, $F \neq E$, out(B,G,A), out(B,H,C), out(E,I,D), out(E,J,F), (B, G) $\cong$ (E, I), (B, H) $\cong$ (E, J), and (G, H) $\cong$ (I, J), it holds that angle(A,B,C) $\cong$ angle(D,E,F) .*

Theorem 169 (11.6) *Asuming that $A \neq B$, and $C \neq B$, it holds that angle(A,B,C) $\cong$ angle(A,B,C) .*

Theorem 170 (11.7) *Asuming that angle(A,B,C) $\cong$ angle(D,E,F) , it holds that angle(D,E,F) $\cong$ angle(A,B,C) .*

Theorem 171 (11.8) *Asuming that angle(A,B,C) $\cong$ angle(D,E,F) , and angle(D,E,F) $\cong$ angle(G,H,I) , it holds that angle(A,B,C) $\cong$ angle(G,H,I) .*

Theorem 172 (11.9) *Asuming that $A \neq B$, and $C \neq B$, it holds that angle(A,B,C) $\cong$ angle(C,B,A) .*

Theorem 173 (11.10) *Asuming that angle(A,B,C) $\cong$ angle(D,E,F) , out(B,G,A), out(B,H,C), out(E,I,D), and out(E,J,F), it holds that angle(G,B,H) $\cong$ angle(I,E,J) .*

Theorem 174 (11.13) *Asuming that angle(A,B,C) $\cong$ angle(D,E,F) , bet(A,B,G), $G \neq B$, bet(D,E,H), and $H \neq E$, it holds that angle(G,B,C) $\cong$ angle(H,E,F) .*

Definition 70 (distinct.1) *Asuming that points A,B,C are distinct in pairs, it holds that $A \neq B$, $A \neq C$ and $B \neq C$.*

Definition 71 (distinct.2) *Asuming that $A \neq B$, $A \neq C$, and $B \neq C$, it holds that points A,B,C are distinct in pairs.*

Theorem 175 (11.14) *Asuming that bet(A,B,D), points A,B,D are distinct in pairs, bet(C,B,E), and points C,B,E are distinct in pairs, it holds that angle(A,B,C) $\cong$ angle(D,B,E) .*

Theorem 176 (11.15.1) *Asuming that $\neg$ col(A,B,C), and $\neg$ col(D,E,F), it holds that there exist a point G such that angle(A,B,C) $\cong$ angle(D,E,G) and points G and F are on the same side of line (E,D).*

Theorem 177 (11.15.2) *Asuming that ¬ col(A,B,C), ¬ col(D,E,F), angle(A,B,C) ≅ angle(D,E,G) , points G and F are on the same side of line (E,D), angle(A,B,C) ≅ angle(D,E,H) , and points H and F are on the same side of line (E,D), it holds that G = H.*

Theorem 178 (11.16) *Asuming that points (A,B,C) form a right angle , A ≠ B, C ≠ B, points (D,E,F) form a right angle , D ≠ E, and F ≠ E, it holds that angle(A,B,C) ≅ angle(D,E,F) .*

Theorem 179 (11.17) *Asuming that points (A,B,C) form a right angle , and angle(A,B,C) ≅ angle(D,E,F) , it holds that points (D,E,F) form a right angle .*

Theorem 180 (11.18.1) *Assuming that bet(A,B,C), points B,A,C are distinct in pairs, D ≠ B, and points (D,B,A) form a right angle , show that angle(D,B,A) ≅ angle(D,B,C) .*

Proof:

1. From the facts that points B,A,C are distinct in pairs, it holds that $B \neq A$, $B \neq C$ and $A \neq C$ (by axiom ax_distinct_1).
2. From the facts that points (D,B,A) form a right angle , it holds that points (A,B,D) form a right angle (by th_8_2).
3. From the facts that bet(A,B,C), it holds that bet(C,B,A) (by th_3_2).
4. From the facts that bet(C,B,A), it holds that col(B,A,C) (by axiom ax_4_10_4).
5. From the facts that points (A,B,D) form a right angle , $A \neq B$, and col(B,A,C), it holds that points (C,B,D) form a right angle (by th_8_3).
6. From the facts that points (C,B,D) form a right angle , it holds that points (D,B,C) form a right angle (by th_8_2).
7. From the fact that $B \neq C$, it holds that $C \neq B$ (by the equality axioms).
8. From the facts that points (D,B,A) form a right angle , $D \neq B$, $A \neq B$, points (D,B,C) form a right angle , $D \neq B$, and $C \neq B$, it holds that angle(D,B,A) ≅ angle(D,B,C) (by th_11_16).

Theorem 181 (11.18.2) *Asuming that bet(C,B,D), points B,C,D are distinct in pairs, A ≠ B, and angle(A,B,C) ≅ angle(A,B,D) , it holds that points (A,B,C) form a right angle .*

Theorem 182 (11.19) *Asuming that points (B,A,C) form a right angle , points (B,A,D) form a right angle , and points C and D are on the same side of line (A,B), it holds that out(A,C,D).*

Theorem 183 (11.21.1.1) *Asuming that out(B,A,C), and angle(A,B,C) ≅ angle(D,E,F) , it holds that out(E,D,F).*

Theorem 184 (11.21.1.2) *Asuming that out(B,A,C), and out(E,D,F), it holds that angle(A,B,C) ≅ angle(D,E,F) .*

Theorem 185 (11.21.2.1) *Asuming that bet(A,B,C), points A,B,C are distinct in pairs, and angle(A,B,C) ≅ angle(D,E,F) , it holds that bet(D,E,F) and points D,E,F are distinct in pairs.*

Theorem 186 (11.21.2.2) *Asuming that bet(A,B,C), points A,B,C are distinct in pairs, bet(D,E,F), and points D,E,F are distinct in pairs, it holds that angle(A,B,C) ≅ angle(D,E,F) .*

Theorem 187 (11.22.1) *Asuming that points A and C are on different sides of line (B,D), points E and G are on different sides of line (F,H), angle(A,B,D) ≅ angle(E,F,H) , and angle(D,B,C) ≅ angle(H,F,G) , it holds that angle(A,B,C) ≅ angle(E,F,G) .*

Theorem 188 (11.22.2) *Asuming that points A and C are on the same side of line (B,D), points E and G are on the same side of line (F,H), angle(A,B,D) ≅ angle(E,F,H) , and angle(D,B,C) ≅ angle(H,F,G) , it holds that angle(A,B,C) ≅ angle(E,F,G) .*

Definition 72 (11.23.1.1) *Asuming that point D is in angle(A,B,C), it holds that there exist a point E such that $A \neq B$, $C \neq B$, $D \neq B$, bet(A,E,C) and $E = B$or and a point E such that $A \neq B$, $C \neq B$, $D \neq B$, bet(A,E,C) and out(B,E,D).*

Definition 73 (11.23.1.2) *Asuming that $A \neq B$, $C \neq B$, $D \neq B$, and bet(A,B,C), it holds that point D is in angle(A,B,C).*

Definition 74 (11.23.2) *Asuming that $A \neq B$, $C \neq B$, $D \neq B$, bet(A,E,C), and out(B,E,D), it holds that point D is in angle(A,B,C).*

Theorem 189 (11.24) *Asuming that point A is in angle(B,C,D), it holds that point A is in angle(D,C,B).*

Theorem 190 (11.25) *Asuming that point A is in angle(B,C,D), out(C,E,B), out(C,F,D), and out(C,G,A), it holds that point G is in angle(E,C,F).*

Definition 75 (11.27.2) *Asuming that point G is in angle(D,E,F), and angle(A,B,C) ≅ angle(D,E,G) , it holds that angle(A,B,C) ≤ angle(D,E,F).*

Definition 76 (11.27.3) *Asuming that angle(A,B,C) ≥(D,E,F), it holds that angle(D,E,F) ≤ angle(A,B,C).*

Definition 77 (11.27.4) *Asuming that angle(D,E,F) ≤ angle(A,B,C), it holds that angle(A,B,C) ≥(D,E,F).*

Definition 78 (11.27.1) *Asuming that angle(A,B,C) ≤ angle(D,E,F), it holds that there exist a point G such that point G is in angle(D,E,F) and angle(A,B,C) ≅ angle(D,E,G) .*

Theorem 191 (11.28) *Asuming that (A, B, C) ≅ (D, E, F), and col(A,C,G), it holds that there exist a point H such that (A, B, C,G) is congruent in pairs with (D, E, F,H).*

Theorem 192 (11.29.1) *Asuming that angle(A,B,C) ≤ angle(D,E,F), it holds that there exist a point G such that point C is in angle(A,B,G) and angle(A,B,G) ≅ angle(D,E,F) .*

Theorem 193 (11.29.2) *Asuming that point C is in angle(A,B,G), and angle(A,B,G) ≅ angle(D,E,F) , it holds that angle(A,B,C) ≤ angle(D,E,F).*

Theorem 194 (11.30) *Asuming that angle(A,B,C) ≤ angle(D,E,F), angle(A,B,C) ≅ angle(G,H,I) , and angle(D,E,F) ≅ angle(J,K,L) , it holds that angle(G,H,I) ≤ angle(J,K,L).*

Theorem 195 (11.31.1) *Asuming that out(B,A,C), D ≠ E, and F ≠ E, it holds that angle(A,B,C) ≤ angle(D,E,F).*

Theorem 196 (11.31.2) *Asuming that A ≠ B, C ≠ B, bet(D,E,F), and points D,E,F are distinct in pairs, it holds that angle(A,B,C) ≤ angle(D,E,F).*

Theorem 197 (11.32) *Assuming that A ≠ B, and C ≠ B, show that angle(A,B,C) ≤ angle(A,B,C).*

Proof:

1. It holds that bet(A,C,C) (by th_3_1).
2. From the facts that $C \neq B$, it holds that out(B,C,C) (by th_6_5).
3. From the facts that $A \neq B$, and $C \neq B$, it holds that angle(A,B,C) ≅ angle(A,B,C) (by th_11_6).
4. From the facts that $A \neq B$, $C \neq B$, $C \neq B$, bet(A,C,C), and out(B,C,C), it holds that point C is in angle(A,B,C) (by axiom ax_11_23_2).
5. From the facts that point C is in angle(A,B,C), and angle(A,B,C) ≅ angle(A,B,C) , it holds that angle(A,B,C) ≤ angle(A,B,C) (by axiom ax_11_27_2).

Theorem 198 (11.33) *Asuming that angle(A,B,C) ≤ angle(D,E,F), and angle(D,E,F) ≤ angle(G,H,I), it holds that angle(A,B,C) ≤ angle(G,H,I).*

Theorem 199 (11.34) *Asuming that angle(A,B,C)* $\leq$ *angle(D,E,F), and angle(D,E,F)* $\leq$ *angle(A,B,C), it holds that angle(A,B,C)* $\cong$ *angle(D,E,F) .*

Theorem 200 (11.35) *Asuming that* $A \neq B$, $C \neq B$, $D \neq E$, *and* $F \neq E$, *it holds that angle(A,B,C)* $\leq$ *angle(D,E,F) or angle(D,E,F)* $\leq$ *angle(A,B,C).*

Theorem 201 (11.36.1) *Asuming that bet(A,B,G), points A,B,G are distinct in pairs, bet(D,E,H), points D,E,H are distinct in pairs, and angle(A,B,C)* $\leq$ *angle(D,E,F), it holds that angle(H,E,F)* $\leq$ *angle(G,B,C).*

Theorem 202 (11.36.2) *Asuming that bet(A,B,G), points A,B,G are distinct in pairs, bet(D,E,H), points D,E,H are distinct in pairs, and angle(H,E,F)* $\leq$ *angle(G,B,C), it holds that angle(A,B,C)* $\leq$ *angle(D,E,F).*

Definition 79 (11.38.1) *Asuming that angle(A,B,C)* $<$ *angle(D,E,F), it holds that angle(A,B,C)* $\leq$ *angle(D,E,F) and angle(A,B,C)* $\ncong$ *angle(D,E,F) .*

Definition 80 (11.38.2) *Asuming that angle(A,B,C)* $\leq$ *angle(D,E,F), and angle(A,B,C)* $\ncong$ *angle(D,E,F) , it holds that angle(A,B,C)* $<$ *angle(D,E,F).*

Definition 81 (11.38.3) *Asuming that angle(A,B,C)* $>$ *angle(D,E,F), it holds that angle(D,E,F)* $<$ *angle(A,B,C).*

Definition 82 (11.38.4) *Asuming that angle(A,B,C)* $<$ *angle(D,E,F), it holds that angle(D,E,F)* $>$ *angle(A,B,C).*

Definition 83 (11.39.1) *Asuming that angle(A,B,C) is an acute angle, it holds that there exist a point D and a point E and a point F such that points (D,E,F) form a right angle and angle(A,B,C)* $<$ *angle(D,E,F).*

Definition 84 (11.39.2) *Asuming that points (D,E,F) form a right angle , and angle(A,B,C)* $<$ *angle(D,E,F), it holds that angle(A,B,C) is an acute angle.*

Definition 85 (11.39.3) *Asuming that angle(A,B,C) is obtuse angle, it holds that there exist a point D and a point E and a point F such that points (D,E,F) form a right angle and angle(A,B,C)* $>$ *angle(D,E,F).*

Definition 86 (11.39.4) *Asuming that points (D,E,F) form a right angle , and angle(A,B,C) > angle(D,E,F), it holds that angle(A,B,C) is obtuse angle.*

Theorem 203 (11.41) *Asuming that ¬ col(A,B,C), bet(B,A,D), and D ≠ A, it holds that angle(A,C,B) < angle(C,A,D) and angle(A,B,C) < angle(C,A,D).*

Theorem 204 (11.43.1) *Asuming that ¬ col(A,B,C), and points (B,A,C) form a right angle , it holds that angle(A,B,C) is an acute angle and angle(A,C,B) is an acute angle.*

Theorem 205 (11.43.2) *Asuming that ¬ col(A,B,C), and angle(B,A,C) is obtuse angle, it holds that angle(A,B,C) is an acute angle and angle(A,C,B) is an acute angle.*

Theorem 206 (11.44.1) *Assuming that not col(A,B,C), and (A, B) ≅ (A, C), show that angle(A,C,B) ≅ angle(A,B,C) .*

Proof:

1. It holds that col(A,A,B) (by th_4_12).
2. It holds that col(A,A,C) (by th_4_12).
3. It holds that col(B,B,A) (by th_4_12).
4. From the facts that col(A,A,B), it holds that col(A,B,A), col(B,A,A), col(B,A,A), col(A,A,B) and col(A,B,A) (by th_4_11).
5. From the facts that col(A,A,C), it holds that col(A,C,A), col(C,A,A), col(C,A,A), col(A,A,C) and col(A,C,A) (by th_4_11).
6. From the facts that col(B,B,A), it holds that col(B,A,B), col(A,B,B), col(A,B,B), col(B,B,A) and col(B,A,B) (by th_4_11).
7. From the facts that (A, B) ≅ (A, C), it holds that (A, B) ≅ (C, A) (by th_2_5).
8. From the facts that (A, B) ≅ (C, A), it holds that (C, A) ≅ (A, B) (by th_2_2).
9. From the facts that (C, A) ≅ (A, B), it holds that (C, A) ≅ (B, A) (by th_2_5).
10. It holds that (B, C) ≅ (C, B) (by axiom ax_1).
11. From the facts that (B, C) ≅ (C, B), it holds that (C, B) ≅ (B, C) (by th_2_2).
    12. It can be trivially proved that $A \neq B$.
        13. From the facts that $A \neq B$, it holds that out(B,A,A) (by th_6_5).
            14. It can be trivially proved that $A \neq C$.
                15. From the facts that $A \neq C$, it holds that out(C,A,A) (by th_6_5).
                    16. It can be trivially proved that $B \neq C$.
                        17. From the facts that $B \neq C$, it holds that out(C,B,B) (by th_6_5).
                        18. From the fact that $B \neq C$, it holds that $C \neq B$ (by the equality axioms).
                        19. From the facts that $C \neq B$, it holds that out(B,C,C) (by th_6_5).
                        20. From the fact that $B \neq C$, it holds that $C \neq B$ (by the equality axioms).
                        21. From the facts that $A \neq C$, $B \neq C$, $A \neq B$, $C \neq B$, out(C,A,A), out(C,B,B), out(B,A,A), out(B,C,C), (C, A) ≅ (B, A), (C, B) ≅ (B, C), and (A, B) ≅ (A, C), it holds that angle(A,C,B) ≅ angle(A,B,C) (by th_11_4_2).

This proves the conjecture.

Theorem 207 (11.44.2) *Asuming that* $\neg$ *col(A,B,C), and angle(A,C,B)* $\cong$ *angle(A,B,C) , it holds that (A, B)* $\cong$ *(A, C).*

Theorem 208 (11.44.3) *Asuming that* $\neg$ *col(A,B,C), and lt(A,B,A,C), it holds that angle(A,C,B)* $<$ *angle(A,B,C).*

Theorem 209 (11.44.4) *Asuming that* $\neg$ *col(A,B,C), and angle(A,C,B)* $<$ *angle(A,B,C), it holds that lt(A,B,A,C).*

Theorem 210 (11.46.1) *Asuming that* $\neg$ *col(A,B,C), and points (B,A,C) form a right angle , it holds that lt(A,B,B,C) and lt(A,C,B,C).*

Theorem 211 (11.46.2) *Asuming that* $\neg$ *col(A,B,C), and angle(B,A,C) is obtuse angle, it holds that lt(A,B,B,C) and lt(A,C,B,C).*

Theorem 212 (11.47) *Asuming that points (A,C,B) form a right angle , and the point D is intersection point of ortogonal lines (C,D) and (A,B), it holds that bet(A,D,B) and points A,D,B are distinct in pairs.*

Theorem 213 (11.49.1) *Asuming that angle(A,B,C)* $\cong$ *angle(D,E,F) , (B, A)* $\cong$ *(E, D), and (B, C)* $\cong$ *(E, F), it holds that (A, C)* $\cong$ *(D, F).*

Theorem 214 (11.49.2) *Asuming that angle(A,B,C)* $\cong$ *angle(D,E,F) , (B, A)* $\cong$ *(E, D), (B, C)* $\cong$ *(E, F), (A, C)* $\cong$ *(D, F), and A* $\neq$ *C, it holds that angle(B,A,C)* $\cong$ *angle(E,D,F) and angle(B,C,A)* $\cong$ *angle(E,F,D) .*

Theorem 215 (11.50.1) *Asuming that* $\neg$ *col(A,B,C), angle(B,A,C)* $\cong$ *angle(E,D,F) , angle(A,B,C)* $\cong$ *angle(D,E,F) , and (A, B)* $\cong$ *(D, E), it holds that (A, C)* $\cong$ *(D, F), (B, C)* $\cong$ *(E, F) and angle(A,C,B)* $\cong$ *angle(D,F,E) .*

Theorem 216 (11.50.2) *Asuming that* $\neg$ *col(A,B,C), angle(B,C,A)* $\cong$ *angle(E,F,D) , angle(A,B,C)* $\cong$ *angle(D,E,F) , and (A, B)* $\cong$ *(D, E), it holds that (A, C)* $\cong$ *(D, F), (B, C)* $\cong$ *(E, F) and angle(B,A,C)* $\cong$ *angle(E,D,F) .*

Theorem 217 (11.51) *Asuming that points A,B,C are distinct in pairs, (A, B) ≅ (D, E), (A, C) ≅ (D, F), and (B, C) ≅ (E, F), it holds that angle(B,A,C) ≅ angle(E,D,F) , angle(A,B,C) ≅ angle(D,E,F) and angle(B,C,A) ≅ angle(E,F,D) .*

Theorem 218 (11.52) *Asuming that angle(A,B,C) ≅ angle(D,E,F) , (A, C) ≅ (D, F), (B, C) ≅ (E, F), and le(B,C,A,C), it holds that (B, A) ≅ (E, D), angle(B,A,C) ≅ angle(E,D,F) and angle(B,C,A) ≅ angle(E,F,D) .*

Theorem 219 (11.53) *Asuming that points (A,D,C) form a right angle , $C \neq D$, points A,B,D are distinct in pairs, and bet(D,A,B), it holds that angle(D,B,C) $<$ angle(D,A,C) and lt(A,C,B,C).*

Theorem 220 (11.57) *Asuming that points B and E are on the same side of line (A,D), points C and F are on the same side of line (A,D), points (B,A,D) form a right angle , points (C,A,D) form a right angle , points (E,D,A) form a right angle , and points (F,D,A) form a right angle , it holds that angle(B,A,C) ≅ angle(E,D,F) .*

Chapter 12

Parallelität (in euklidischen Sinne) (Parallelism (Euclidean Sense))

Axiom 11 (int1) *Asuming that lines (A,B) and (C,D) are intersecting, it holds that there exist a point E such that point E is the intersection of line(A,B) and line(C,D).*

Axiom 12 (int2) *Asuming that point E is the intersection of line(A,B) and line(C,D), it holds that lines (A,B) and (C,D) are intersecting.*

Definition 87 (12.2.1) *Asuming that $A \neq B$, $C \neq D$, and lines (A,B) and (C,D) are not intersecting, it holds that lines (A,B) and (C,D) are parallel.*

Definition 88 (12.2.2) *Asuming that lines (A,B) and (C,D) are parallel, it holds that $A \neq B$, $C \neq D$ and lines (A,B) and (C,D) are not intersecting.*

Definition 89 (12.3.1) *Asuming that lines (A,B) and (C,D) are parallel or are the same lines, it holds that $A \neq B$, $C \neq D$ and lines (A,B) and (C,D) are parallelor $A \neq B$, $C \neq D$ and line(A,B)=line(C,D).*

Definition 90 (12.3.2) *Asuming that $A \neq B$, $C \neq D$, and lines (A,B) and (C,D) are parallel, it holds that lines (A,B) and (C,D) are parallel or are the same lines.*

Definition 91 (12.3.3) *Asuming that $A \neq B$, $C \neq D$, and line(A,B)=line(C,D), it holds that lines (A,B) and (C,D) are parallel or are the same lines.*

Theorem 221 (12.4) *Assuming that $A \neq B$, show that lines (A,B) and (A,B) are parallel or are the same lines.*

Proof:

1. From the facts that $A \neq B$, it holds that $A \in$ line(A,B), $B \in$ line(A,B) and line(A,B)=line(B,A) (by th_6_17).
2. From the fact that $A \neq B$, it holds that $B \neq A$ (by the equality axioms).
3. From the facts that $A \neq B$, $B \neq A$, and $B \in$ line(A,B), it holds that line(A,B)=line(A,B) (by th_6_16).
4. From the facts that $A \neq B$, $A \neq B$, and line(A,B)=line(A,B), it holds that lines (A,B) and (A,B) are parallel or are the same lines (by axiom ax_12_3_3).

Theorem 222 (12.5.1) *Asuming that lines (A,B) and (C,D) are parallel, it holds that lines (C,D) and (A,B) are parallel.*

Theorem 223 (12.5.2) *Asuming that lines (A,B) and (C,D) are parallel or are the same lines, it holds that lines (C,D) and (A,B) are parallel or are the same lines.*

Theorem 224 (12.6) *Asuming that lines (A,B) and (C,D) are parallel, $E \in$ line(C,D), and $F \in$ line(C,D), it holds that points E and F are on the same side of line (A,B).*

Theorem 225 (12.9) *Assuming that lines (A,B), (C,D) and (E,F) are coplanar, lines (A,B) and (E,F) are ortogonal, and lines (C,D) and (E,F) are ortogonal, show that lines (A,B) and (C,D) are parallel or are the same lines.*

Theorem 226 (12.10) *Asuming that $A \neq B$, it holds that there exist a point D and a point E such that $D \neq E$, lines (A,B) and (D,E) are parallel or are the same lines and $C \in$ line(D,E).*

Theorem 227 (12.11) *Asuming that $A \neq B$, $C \notin$ line(A,B), lines (A,B) and (D,E) are parallel or are the same lines, lines (A,B) and (F,G) are parallel or are the same lines, $C \in$ line(D,E), and $C \in$ line(F,G), it holds that line(D,E)=line(F,G).*

Theorem 228 (12.15) *Asuming that lines (A,B) and (C,D) are parallel or are the same lines, and lines (C,D) and (E,F) are parallel or are the same lines, it holds that lines (A,B) and (E,F) are parallel or are the same lines.*

Theorem 229 (12.16) *Asuming that lines (A,B) and (C,D) are parallel or are the same lines, and point G is the intersection of line(E,F) and line(A,B), it holds that there exist a point H such that point H is the intersection of line(E,F) and line(C,D).*

Theorem 230 (12.17) *Asuming that the point E is midpoint of points A and C, the point E is midpoint of points B and D, and $A \neq B$, it holds that lines (A,B) and (C,D) are parallel or are the same lines.*

Theorem 231 (12.18) *Asuming that (A, B) $\cong$ (C, D), (B, C) $\cong$ (D, A), $\neg$ col(A,B,C), $B \neq D$, col(A,E,C), and col(B,E,D), it holds that lines (A,B) and (C,D) are parallel or are the same lines, lines (B,C) and (D,A) are parallel or are the same lines, points B and D are on different sides of line (A,C) and points A and C are on different sides of line (B,D).*

Theorem 232 (12.19) *Asuming that $\neg$ col(A,B,C), lines (A,B) and (C,D) are parallel or are the same lines, and lines (B,C) and (D,A) are parallel or are the same lines, it holds that (A, B) $\cong$ (C, D), (B, C) $\cong$ (D, A), points B and D are on different sides of line (A,C) and points A and C are on different sides of line (B,D).*

Theorem 233 (12.20) *Asuming that lines (A,B) and (C,D) are parallel or are the same lines, (A, B) ≅ (C, D), and points B and D are on different sides of line (A,C), it holds that lines (B,C) and (D,A) are parallel or are the same lines, (B, C) ≅ (D, A) and points A and C are on different sides of line (B,D).*

Theorem 234 (12.21.1) *Asuming that points B and D are on different sides of line (A,C), and lines (A,B) and (C,D) are parallel or are the same lines, it holds that angle(B,A,C) ≅ angle(D,C,A) .*

Theorem 235 (12.21.2) *Asuming that points B and D are on different sides of line (A,C), and angle(B,A,C) ≅ angle(D,C,A) , it holds that lines (A,B) and (C,D) are parallel or are the same lines.*

Theorem 236 (12.22.1) *Asuming that out(E,A,C), points B and D are on the same side of line (E,A), and lines (A,B) and (C,D) are parallel or are the same lines, it holds that angle(B,A,E) ≅ angle(D,C,E) .*

Theorem 237 (12.22.2) *Asuming that out(E,A,C), points B and D are on the same side of line (E,A), and angle(B,A,E) ≅ angle(D,C,E) , it holds that lines (A,B) and (C,D) are parallel or are the same lines.*

Theorem 238 (12.23) *Asuming that ¬ col(A,B,C), it holds that there exist a point D and a point E such that points B and D are on different sides of line (A,C), points C and E are on different sides of line (A,B), bet(D,A,E), angle(A,B,C) ≅ angle(B,A,E) and angle(A,C,B) ≅ angle(C,A,D) .*

Chapter 13

Die Sätze von Papus-Pascal und von Desargues (Theorems of Pappus-Pascal, and Desargues)

Chapter 14

Einführung eines angeordneten Körpers (Introduction to Ordered Fields)

Chapter 15

Längen von Strecken (Lengths of Segments)

Chapter 16

Koordinaten (Coordinates)