Automated Generation of Formal and Readable Proofs in Geometry Using Coherent Logic - Appendix

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A Axiom Systems

A.1 Signature

Sorts:

point line plane

Primitive predicate symbols:

$inc\_po\_l(point, line)$ stands for "the point ... is incident with the line ... "
$inc\_po\_pl(point, plane)$ stands for "the point ... is incident with the plane ... "
$inc\_l\_pl(line, plane)$ stands for "the line ... is incident with the plane ... "
$bet(point, point, point)$ stands for "the point ... is between points ... and ... "
$cong(point, point, point, point)$ stands for "the segment determined by the first two points is congruent to the segment determined by the second two"

Defined predicate symbols:

$int\_l\_l(line, line)$ stands for "the lines ... and ... intersect"
$int\_l\_pl(line, plane)$ stands for "the line ... and plane ... intersect"
$int\_pl\_pl(plane, plane)$ stands for "the planes ... and ... intersect"
$col(point, point, point)$ stands for "the three points are collinear"
$comp(point, point, point, point)$ stands for "the four points are coplanar"
$cong\_angle(point, point, point, point, point, point$ stands for "the angle determined by first three points is congruent to the angle determined by the latter three points"

In the following formulations, for brevity, instead of inc_p_l we write just inc (as the type is always clear from the context), instead of $eq\_point(A, B)$ and $\neg eq\_point(A, B)$ we write $A = B$ and $A \neq B$, etc.

A.2 Hilbert's axiom system

Definition D1: $\forall A : point\ \forall B : point\ \forall C : point\ \forall p : line\ (inc(A,p)\ \wedge\ inc(B,p)\ \wedge\ inc(C,p)\ \Rightarrow\ col(A,B,C))$

Definition D2: $\forall A : point\ \forall B : point\ \forall C : point\ (col(A,B,C)\ \Rightarrow\ \exists p : line\ (inc(A,p)\ \wedge\ inc(B,p)\ \wedge\ inc(C,p)))$

Definition D3: $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ \forall \alpha : plane\ (inc(A,\alpha)\ \wedge\ inc(B,\alpha)\ \wedge\ inc(C,\alpha)\ \wedge\ inc(D,\alpha)\ \Rightarrow\ comp(A,B,C,D))$

Definition D4: $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ (comp(A,B,C,D)\ \Rightarrow\ \exists \alpha : plane\ (inc(A,\alpha)\ \wedge\ inc(B,\alpha)\ \wedge\ inc(C,\alpha)\ \wedge\ inc(D,\alpha)))$

Definition D5: $\forall p : line\ \forall q : line\ \forall A : point\ (p \neq q\ \wedge\ inc(A,p)\ \wedge\ inc(A,q)\ \Rightarrow\ int(p,q))$

Definition D6: $\forall p : line\ \forall q : line(int(p,q)\ \Rightarrow\ \exists A : point\ (inc(A,p)\ \wedge\ inc(A,q)\ \wedge\ p \neq q))$

Definition D7: $\forall \alpha : plane\ \forall \beta : plane\ \forall A : point(\alpha \neq \beta\ \wedge\ inc(A,\alpha)\ \wedge\ inc(A,\beta)\ \Rightarrow\ int(\alpha,\beta))$

Definition D8: $\forall \alpha : plane\ \forall \beta : plane(int(\alpha,\beta)\ \Rightarrow\ \exists A : point\ (inc(A,\alpha)\ \wedge\ inc(A,\beta)\ \wedge\ \alpha \neq \beta))$

Definition D9: $\forall A : point\ \forall p : line\ \forall \alpha : plane\ (inc(A,p)\ \wedge\ inc(A,\alpha)\ \wedge\ \neg inc(p,\alpha)\ \Rightarrow\ int(p,\alpha))$

Definition D10: $\forall p : line\ \forall \alpha : plane\ (int(p,\alpha)\ \Rightarrow\ \exists A : point\ (inc(A,p)\ \wedge\ inc(A,\alpha)\ \wedge\ \neg inc(p,\alpha)))$

Definition D11: $\forall A : point\ \forall B : point\ \forall C : point\ \forall X : point\ \forall Y : point\ \forall Z : point\ \forall U : point\ \forall V : point\ (A \neq B\ \wedge\ A \neq C\ \wedge\ X \neq Y\ \wedge\ X \neq Z\ \wedge\ col(X,Y,U)\ \wedge\ \neg bet(U,X,Y)\ \wedge\ col(X,Z,V)\ \wedge\ \neg bet(V,X,Z)\ \wedge\ cong(A,B,X,U)\ \wedge\ cong(A,C,X,V)\ \wedge\ cong(B,C,U,V)\ \Rightarrow\ cong\_angle(A,B,C,X,Y,Z))$

Definition D12: $\forall A : point\ \forall B : point\ \forall C : point\ \forall X : point\ \forall Y : point\ \forall Z : point\ (cong\_angle(A,B,C,X,Y,Z)\ \Rightarrow\ \exists U : point\ \exists V : point\ (A \neq B\ \wedge\ A \neq C\ \wedge\ X \neq Y\ \wedge\ X \neq Z\ \wedge\ col(X,Y,U)\ \wedge\ \neg bet(U,X,Y)\ \wedge\ col(X,Z,V)\ \wedge\ \neg bet(V,X,Z)\ \wedge\ cong(A,B,X,U)\ \wedge\ cong(A,C,X,V)\ \wedge\ cong(B,C,U,V)))$

Axiom I1: $\forall A : point\ \ \forall B : point\ \ (A \neq B \ \Rightarrow\ \exists p : line\ (inc(A,p)\ \wedge\ inc(B,p)))$

Axiom I2: $\forall A : point\ \ \forall B : point\ \ \forall p : line\ \ \forall q : line\ \ (inc(A,p)\ \wedge\ inc(A,q)\ \wedge\ inc(B,p)\ \wedge\ inc(B,q)\ \wedge\ A \neq B\ \Rightarrow\ p = q)$

Axiom I3a: $\forall p : line\ \ \exists A : point\ \ \exists B : point\ (inc(A,p)\ \wedge\ inc(B,p)\ \wedge\ A \neq B)$

Axiom I3b: $\exists A : point\ \ \exists B : point\ \ \exists C : point\ (\neg col(A,B,C))$

Axiom I4a: $\forall A : point\ \ \forall B : point\ \ \forall C : point\ \ (A \neq B\ \wedge\ A \neq C\ \wedge\ B \neq C\ \wedge\ \neg col(A,B,C)\ \Rightarrow\ \exists \alpha : plane\ (inc(A,\alpha)\ \wedge\ inc(B,\alpha)\ \wedge\ inc(C,\alpha)))$

Axiom I4b: $\forall \alpha : plane\ \ \exists A : point\ (inc(A,\alpha))$

Axiom I5: $\forall \alpha : plane\ \ \forall \beta : plane\ \ \forall A : point\ \ \forall B : point\ \ \forall C : point\ (A \neq B\ \wedge\ A \neq C\ \wedge\ B \neq C\ \wedge\ inc(A,\alpha)\ \wedge\ inc(A,\beta)\ \wedge\ inc(B,\alpha)\ \wedge\ inc(B,\beta)\ \wedge\ inc(C,\alpha)\ \wedge\ inc(C,\beta)\ \wedge\ \neg col(A,B,C)\ \Rightarrow\ \alpha = \beta)$

Axiom I6: $\forall \alpha : plane\ \ \forall p : line\ \ \forall A : point\ \ \forall B : point\ (inc(A,\alpha)\ \wedge\ inc(A,p)\ \wedge\ inc(B,\alpha)\ \wedge\ inc(B,p)\ \wedge\ A \neq B\ \Rightarrow\ inc(p,\alpha))$

Axiom I7: $\forall \alpha : plane\ \ \forall \beta : plane\ \ \forall A : point\ \ (\alpha \neq \beta\ \wedge\ inc(A,\alpha)\ \wedge\ inc(A,\beta)\ \Rightarrow\ \exists B : point\ (A \neq B\ \wedge\ inc(B,\alpha)\ \wedge\ inc(B,\beta)))$

Axiom I8: $\exists A : point\ \ \exists B : point\ \ \exists C : point\ \ \exists D : point\ (\neg comp(A,B,C,D))$

Axiom II1: $\forall A : point\ \ \forall B : point\ \ \forall C : point\ (bet(A,B,C)\ \Rightarrow\ col(A,B,C)\ \wedge\ A \neq B\ \wedge\ A \neq C\ \wedge\ B \neq C\ \wedge\ bet(C,B,A))$

Axiom II2: $\forall A : point\ \ \forall B : point\ \ (A \neq B\ \Rightarrow\ \exists C : point\ (bet(A,B,C)))$

Axiom II3: $\forall A : point\ \ \forall B : point\ \ \forall C : point\ \ (A \neq B\ \wedge\ A \neq C\ \wedge\ B \neq C\ \wedge\ col(A,B,C)\ \Rightarrow\ bet(A,B,C)\ \vee\ bet(B,C,A)\ \vee\ bet(C,A,B))$

Axiom II4: $\forall A : point\ \ \forall B : point\ \ \forall C : point\ \ \forall D : point\ \ \forall \alpha : plane\ \ \forall p : line\ \ (A \neq B\ \wedge\ A \neq C\ \wedge\ B \neq C\ \wedge\ \neg col(A,B,C)\ \wedge\ inc(A,\alpha)\ \wedge\ inc(B,\alpha)\ \wedge\ inc(C,\alpha)\ \wedge\ inc(p,\alpha)\ \wedge\ \neg inc(A,p)\ \wedge\ inc(D,p)\ \wedge\ bet(B,D,C)\ \Rightarrow\ (\exists E : point\ ((inc(E,p)\ \wedge\ bet(C,E,A)))\ \vee\ (\exists F : point\ \ (inc(F,p)\ \wedge\ bet(A,F,B))))$

Axiom III1: $\forall A : point\ \ \forall B : point\ \ \forall C : point\ \ \forall D : point\ \ (A \neq B\ \wedge\ C \neq D\ \Rightarrow\ \exists E : point\ (bet(B,A,E)\ \wedge\ cong(A,E,C,D)))$

Axiom III2: $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ \forall E : point\ \forall F : point\ (cong(A,B,C,D)\ \wedge\ cong(A,B,E,F)\ \Rightarrow\ cong(C,D,E,F)$

Axiom III3: $\forall A : point\ \forall B : point\ \forall C : point\ \forall A_1 : point\ \forall B_1 : point\ \forall C_1 : point\ (bet(A,B,C)\ \wedge\ bet(A_1,B_1,C_1)\ \wedge\ cong(A,B,A_1,B_1)\ \wedge\ cong(B,C,B_1,C_1)\ \Rightarrow\ cong(A,C,A_1,C_1))$

Axiom III4_a: $\forall A : point\ \forall B : point\ \forall C : point\ (A \neq B\ \wedge\ A \neq C\ \Rightarrow\ cong\_angle(A,B,C,A,B,C))$

Axiom III4_b1: $\forall A : point\ \forall B : point\ \forall C : point\ \forall P : point\ \forall Q : point\ \forall M : point\ (A \neq B\ \wedge\ A \neq C\ \wedge\ P \neq Q\ \wedge\ \neg col(P,Q,M)\ \Rightarrow\ \exists l : line\ \exists R_1 : point\ \exists R_2 : point\ \exists N_1 : point\ \exists N_2 : point\ (cong\_angle(A,B,C,P,Q,R_1)\ \wedge\ cong\_angle(A,B,C,P,Q,R_2)\ \wedge\ inc(P,l)\ \wedge\ inc(Q,l)\ \wedge\ bet(M,N_1,R_1)\ \wedge\ inc(N_1,l)\ \wedge\ bet(R_2,N_2,R_1)\ \wedge\ inc(N_2,l)))$

Axiom III4_b2: $\forall A : point\ \forall B : point\ \forall C : point\ \forall P : point\ \forall M : point\ \forall R_1 : point\ \forall R_2 : point\ \forall N_1 : point\ \forall N_2 : point\ \forall l : line\ (A \neq B\ \wedge\ A \neq C\ \wedge\ P \neq Q\ \wedge\ \neg col(P,Q,M)\ \wedge\ cong\_angle(A,B,C,P,Q,R_1)\ \wedge\ cong\_angle(A,B,C,P,Q,R_2)\ \wedge\ inc(P,l)\ \wedge\ inc(Q,l)\ \wedge\ bet(M,N_1,R_1)\ \wedge\ inc(N_1,l)\ \wedge\ bet(M,N_2,R_2)\ \wedge\ inc(N_2,l)\ \Rightarrow\ bet(P,R_1,R_2)\ \vee\ bet(P,R_2,R_1)\ \vee\ R_1 = R_2)$

Axiom III5: $\forall A : point\ \forall B : point\ \forall C : point\ \forall X : point\ \forall Y : point\ \forall Z : point\ (cong(A,B,X,Y)\ \wedge\ cong(A,C,X,Z)\ \wedge\ cong\_angle(A,B,C,X,Y,Z)\ \Rightarrow\ cong\_angle(B,A,C,Y,X,Z))$

Axiom IV_a: $\forall A : point\ \forall p : line\ \forall \alpha : plane\ (\neg inc(A,p)\ \wedge\ inc(A,\alpha)\ \wedge\ inc(p,\alpha)\ \Rightarrow\ \exists q : line\ (inc(A,q)\ \wedge\ inc(q,\alpha)\ \wedge\ \neg int(q,p)))$

Axiom IV_b: $\forall A : point\ \forall p : line\ \forall \alpha : plane\ \forall q : line\ \forall r : line\ (\neg inc(A,p)\ \wedge\ inc(A,\alpha)\ \wedge\ inc(p,\alpha)\ \wedge\ inc(A,q)\ \wedge\ inc(q,\alpha)\ \wedge\ \neg int(p,q)\ \wedge\ inc(A,r)\ \wedge\ inc(r,\alpha)\ \wedge\ \neg int(p,r)\ \Rightarrow\ q = r)$

A.3 Borsuk's axiom system

Definition D1: $\forall A : point\ \forall B : point\ \forall C : point\ \forall p : line\ (inc(A,p)\ \wedge\ inc(B,p)\ \wedge\ inc(C,p)\ \Rightarrow\ col(A,B,C))$

Definition D2: $\forall A : point\ \forall B : point\ \forall C : point\ (col(A,B,C)\ \Rightarrow\ \exists p : line\ (inc(A,p)\ \wedge\ inc(B,p)\ \wedge\ inc(C,p)))$

Definition D3: $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ \forall \alpha : plane\ (inc(A,\alpha)\ \wedge\ inc(B,\alpha)\ \wedge\ inc(C,\alpha)\ \wedge\ inc(D,\alpha)\ \Rightarrow\ comp(A,B,C,D))$

Definition D4: $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ (comp(A,B,C,D)\ \Rightarrow\ \exists \alpha : plane\ (inc(A,\alpha)\ \wedge\ inc(B,\alpha)\ \wedge\ inc(C,\alpha)\ \wedge\ inc(D,\alpha)))$

Definition D5: $\forall p : line\ \forall q : line\ \forall A : point\ (p \neq q\ \wedge\ inc(A,p)\ \wedge\ inc(A,q)\ \Rightarrow\ int(p,q))$

Definition D6: $\forall p : line\ \forall q : line(int(p,q)\ \Rightarrow\ \exists A : point\ (inc(A,p)\ \wedge\ inc(A,q)\ \wedge\ p \neq q))$

Definition D7: $\forall \alpha : plane\ \forall \beta : plane\ \forall A : point(\alpha \neq \beta\ \wedge\ inc(A,\alpha)\ \wedge\ inc(A,\beta)\ \Rightarrow\ int(\alpha,\beta))$

Definition D8: $\forall \alpha : plane\ \forall \beta : plane(int(\alpha,\beta)\ \Rightarrow\ \exists A : point\ (inc(A,\alpha)\ \wedge\ inc(A,\beta)\ \wedge\ \alpha \neq \beta))$

Definition D9: $\forall A : point\ \forall p : line\ \forall \alpha : plane\ (inc(A,p)\ \wedge\ inc(A,\alpha)\ \wedge\ \neg inc(p,\alpha)\ \Rightarrow\ int(p,\alpha))$

Definition D10: $\forall p : line\ \forall \alpha : plane\ (int(p,\alpha)\ \Rightarrow\ \exists A : point\ (inc(A,p)\ \wedge\ inc(A,\alpha)\ \wedge\ \neg inc(p,\alpha)))$

Axiom I1 : $\forall p : line\ \exists A : point\ \exists B : point\ (inc(A,p)\ \wedge\ A \neq B\ \wedge\ inc(B,p))$

Axiom I2 : $\forall A : point\ \forall B : point\ (A \neq B\ \Rightarrow\ \exists p : line\ (inc(A,p)\ \wedge\ inc(B,p)))$

Axiom I3 : $\forall A : point\ \forall B : point\ \forall p : line\ \forall q : line\ (inc(A,p)\ \wedge\ inc(A,q)\ \wedge\ inc(B,p)\ \wedge\ inc(B,q)\ \wedge\ A \neq B\ \Rightarrow\ p = q)$

Axiom I4 : $\forall \alpha : plane\ \exists A : point\ \exists B : point\ \exists C : point\ (inc(A,\alpha)\ \wedge\ inc(B,\alpha)\ \wedge\ inc(C,\alpha)\ \wedge\ \neg col(A,B,C))$

Axiom I5 : $\forall A : point\ \forall B : point\ \forall C : point\ \exists \alpha : plane\ (inc(A, \alpha) \wedge inc(B, \alpha) \wedge inc(C, \alpha))$

Axiom I6 : $\forall \alpha : plane\ \forall \beta : plane\ \forall A : point\ \forall B : point\ \forall C : point (inc(A, \alpha) \wedge inc(A, \beta) \wedge inc(B, \alpha) \wedge inc(B, \beta) \wedge inc(C, \alpha) \wedge inc(C, \beta) \wedge \neg col(A, B, C) \Rightarrow \alpha = \beta)$

Axiom I7 : $\forall \alpha : plane\ \forall p : line\ \forall A : point\ \forall B : point\ (inc(A, \alpha) \wedge inc(A, p) \wedge inc(B, \alpha) \wedge inc(B, p) \wedge A \neq B \Rightarrow inc(p, \alpha))$

Axiom I8 : $\forall \alpha : plane\ \forall \beta : plane\ \forall A : point(\alpha \neq \beta \wedge inc(A, \alpha) \wedge inc(A, \beta) \Rightarrow \exists B : point\ (A \neq B \wedge inc(B, \alpha) \wedge inc(B, \beta)))$

Axiom I9 : $\exists A : point\ \exists B : point\ \exists C : point\ \exists D : point\ (\neg comp(A, B, C, D))$

Axiom O1 : $\forall A : point\ \forall B : point\ \forall C : point\ (bet(A, B, C) \Rightarrow col(A, B, C) \wedge A \neq B \wedge A \neq C \wedge B \neq C)$

Axiom O2 : $\forall A : point\ \forall B : point\ \forall C : point\ (bet(A, B, C) \Rightarrow bet(C, B, A))$

Axiom O3 : $\forall A : point\ \forall B : point\ \forall C : point\ (bet(A, B, C) \Rightarrow \neg bet(A, C, B))$

Axiom O4 : $\forall A : point\ \forall B : point\ \forall C : point\ (A \neq B \wedge A \neq C \wedge B \neq C \wedge col(A, B, C) \Rightarrow bet(A, B, C) \vee bet(B, C, A) \vee bet(C, A, B))$

Axiom O5 : $\forall A : point\ \forall B : point\ (A \neq B \Rightarrow \exists C : point\ (bet(A, B, C)))$

Axiom O6: $\forall A : point\ \forall B : point\ (A \neq B \Rightarrow \exists C : point\ (bet(A, C, B)))$

Axiom O7: $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ (bet(A, B, C) \wedge bet(B, C, D) \Rightarrow bet(A, C, D))$

Axiom O8: $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ (bet(A, B, D) \wedge bet(B, C, D) \Rightarrow bet(A, B, C))$

Axiom O9 : $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ \forall \alpha : plane\ \forall p : line\ (\neg col(A, B, C) \wedge inc(A, \alpha) \wedge inc(B, \alpha) \wedge inc(C, \alpha) \wedge inc(p, \alpha) \wedge \neg inc(A, p) \wedge inc(D, p) \wedge bet(B, D, C) \Rightarrow (\exists E : point\ ((inc(E, p) \wedge bet(C, E, A))) \vee (\exists F : point\ (inc(F, p) \wedge bet(A, F, B)))))$

Axiom C1 : $\forall A : point\ \forall B : point\ \forall C : point\ (cong(A, A, B, C) \Rightarrow B = C)$

Axiom C2 : $\forall A : point\ \forall B : point\ \Rightarrow cong(A, B, B, A)$

Axiom C3 : $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ \forall E : point\ \forall F : point\ (cong(A,B,C,D) \wedge cong(A,B,E,F) \Rightarrow cong(C,D,E,F))$

Axiom C4: $\forall A : point\ \forall B : point\ \forall C : point\ \forall A_1 : point\ \forall B_1 : point\ \forall C_1 : point\ (bet(A,B,C) \wedge bet(A_1,B_1,C_1) \wedge cong(A,B,A_1,B_1) \wedge cong(B,C,B_1,C_1) \Rightarrow cong(A,C,A_1,C_1))$

Axiom C5: $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ (A \neq B \wedge C \neq D \Rightarrow \exists E : point\ (bet(B,A,E) \wedge cong(A,E,C,D)))$

Axiom C5_1: $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ \forall E : point\ \forall F : point\ (A \neq B \wedge C \neq D \wedge bet(B,A,E) \wedge cong(A,E,C,D) \wedge bet(B,A,F) \wedge cong(A,F,C,D) \Rightarrow E = F)$

Axiom C6 : $\forall p : line\ \forall q : line\ \forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ \forall E : point\ \forall F : point\ \forall G : point\ \forall H : point\ (inc(A,p) \wedge inc(B,p) \wedge inc(C,p) \wedge \neg inc(D,p) \wedge inc(E,q) \wedge inc(F,q) \wedge inc(G,q) \wedge \neg inc(H,q) \wedge bet(A,B,C) \wedge bet(E,F,G) \wedge cong(A,B,E,F) \wedge cong(B,C,F,G) \wedge cong(D,A,H,E) \wedge cong(D,B,H,F) \Rightarrow (cong(D,C,H,G)))$

Axiom C7: $\forall A : point\ \forall B : point\ \forall P : point\ \forall Q : point\ \forall R : point\ \forall D : point\ (A \neq B \wedge \neg col(P,Q,R) \wedge \neg col(A,B,D) \wedge cong(A,B,P,Q) \Rightarrow \exists C : point\ \exists E : point\ (bet(D,E,C) \wedge col(A,B,E) \wedge cong(A,C,P,R) \wedge cong(B,C,D,R)))$

Axiom C7_1: $\forall A : point\ \forall B : point\ \forall P : point\ \forall Q : point\ \forall R : point\ \forall D : point\ \forall C_1 : point\ \forall C_2 : point\ \forall E_1 : point\ \forall E_2 : point\ (A \neq B \wedge \neg col(P,Q,R) \wedge \neg col(A,B,D) \wedge cong(A,B,P,Q) \wedge bet(D,E_1,C_1) \wedge col(A,B,E_1) \wedge cong(A,C_1,P,R) \wedge cong(B,C_1,Q,R) \wedge bet(D,E_2,C_2) \wedge col(A,B,E_2) \wedge cong(A,C_2,P,R) \wedge cong(B,C_2,Q,R) \Rightarrow C_1 = C_2)$

Axiom E1 : $\forall A : point\ \forall p : line\ \forall \alpha : plane(\neg inc(A,p) \wedge inc(A,\alpha) \wedge inc(p,\alpha) \Rightarrow \exists q : line\ (inc(A,q) \wedge inc(q,\alpha) \wedge \neg int(q,p)))$

Axiom E2 : $\forall A : point\ \forall p : line\ \forall \alpha : plane\ \forall q : line\ \forall r : line\ (inc(A,\alpha) \wedge inc(p,\alpha) \wedge \neg inc(A,p) \wedge inc(A,q) \wedge inc(q,\alpha) \wedge \neg int(p,q) \wedge inc(A,r) \wedge inc(r,\alpha) \wedge \neg int(p,r) \Rightarrow q = r)$

A.4 ARGO axiom system

Definition D1: $\forall A : point\ \forall B : point\ \forall C : point\ \forall p : line\ (inc(A, p)\ \wedge\ inc(B, p)\ \wedge\ inc(C, p)\ \Rightarrow\ col(A, B, C))$

Definition D2: $\forall A : point\ \forall B : point\ \forall C : point\ (col(A, B, C)\ \Rightarrow\ \exists p : line\ (inc(A, p)\ \wedge\ inc(B, p)\ \wedge\ inc(C, p)))$

Definition D3: $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ \forall \alpha : plane\ (inc(A, \alpha)\ \wedge\ inc(B, \alpha)\ \wedge\ inc(C, \alpha)\ \wedge\ inc(D, \alpha)\ \Rightarrow\ comp(A, B, C, D))$

Definition D4: $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ (comp(A, B, C, D)\ \Rightarrow\ \exists \alpha : plane\ (inc(A, \alpha)\ \wedge\ inc(B, \alpha)\ \wedge\ inc(C, \alpha)\ \wedge\ inc(D, \alpha)))$

Definition D5: $\forall p : line\ \forall q : line\ \forall A : point\ (p \neq q\ \wedge\ inc(A, p)\ \wedge\ inc(A, q)\ \Rightarrow\ int(p, q))$

Definition D6: $\forall p : line\ \forall q : line(int(p, q)\ \Rightarrow\ \exists A : point\ (inc(A, p)\ \wedge\ inc(A, q)\ \wedge\ p \neq q))$

Definition D7: $\forall \alpha : plane\ \forall \beta : plane\ \forall A : point(\alpha \neq \beta\ \wedge\ inc(A, \alpha)\ \wedge\ inc(A, \beta)\ \Rightarrow\ int(\alpha, \beta))$

Definition D8: $\forall \alpha : plane\ \forall \beta : plane(int(\alpha, \beta)\ \Rightarrow\ \exists A : point\ (inc(A, \alpha)\ \wedge\ inc(A, \beta)\ \wedge\ \alpha \neq \beta))$

Definition D9: $\forall A : point\ \forall p : line\ \forall \alpha : plane\ (inc(A, p)\ \wedge\ inc(A, \alpha)\ \wedge\ \neg inc(p, \alpha)\ \Rightarrow\ int(p, \alpha))$

Definition D10: $\forall p : line\ \forall \alpha : plane\ (int(p, \alpha)\ \Rightarrow\ \exists A : point\ (inc(A, p)\ \wedge\ inc(A, \alpha)\ \wedge\ \neg inc(p, \alpha)))$

Axiom N1: $\forall A : point\ \forall p : line\ \forall \alpha : plane\ (inc(A, p)\ \wedge\ inc(p, \alpha)\ \Rightarrow\ inc(A, \alpha))$

Axiom N2_1: $\forall A : point\ \forall p : line\ \forall \alpha : plane\ (\neg inc(A, p)\ \wedge\ inc(A, \alpha)\ \wedge\ inc(p, \alpha)\ \Rightarrow\ \exists q : line\ (inc(A, q)\ \wedge\ inc(q, \alpha)\ \wedge\ \neg int(q, p)))$

Axiom N2_2: $\forall \alpha : plane\ \forall A : point\ \forall p : line\ \forall q : line\ \forall r : line\ (inc(A, \alpha)\ \wedge\ inc(p, \alpha)\ \wedge\ \neg inc(A, p)\ \wedge\ inc(A, q)\ \wedge\ inc(q, \alpha)\ \wedge\ \neg int(p, q)\ \wedge\ inc(A, r)\ \wedge\ inc(r, \alpha)\ \wedge\ \neg int(p, r)\ \Rightarrow\ q = r)$

Axiom N3: $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ (cong(A, B, C, D)\ \wedge\ A = B\ \Rightarrow\ C = D)$

Axiom N4: $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ \forall E : point\ \forall F : point\ (cong(A, B, C, D) \wedge cong(A, B, E, F) \ \Rightarrow\ cong(C, D, E, F))$

Axiom N5: $\forall A_1 : point\ \forall B_1 : point\ \forall C_1 : point\ \forall A_2 : point\ \forall B_2 : point\ \forall C_2 : point\ (bet(A_1, C_1, B_1)\ \wedge\ bet(A_2, C_2, B_2)\ \wedge\ cong(A_1, C_1, A_2, C_2)\ \wedge\ cong(B_1, C_1, B_2, C_2)\ \Rightarrow\ cong(A_1, B_1, A_2, B_2))$

Axiom N6: $\forall A : point\ \forall B : point\ \forall C : point\ (bet(A, B, C)\ \Rightarrow\ col(A, B, C)\ \wedge\ A \neq B\ \wedge\ A \neq C\ \wedge\ B \neq C)$

Axiom N7: $\forall A : point\ \forall B : point\ \forall C : point\ (bet(A, B, C)\ \Rightarrow\ bet(C, B, A))$

Axiom N8: $\forall A : point\ \forall B : point\ \forall C : point\ (bet(A, B, C)\ \Rightarrow\ \neg bet(A, C, B))$

Axiom N9: $\forall A : point\ \forall B : point\ \Rightarrow\ cong(A, B, B, A)$

Axiom N10: $\forall A : point\ \forall B : point\ \forall p : line\ \forall q : line\ (inc(A, p)\ \wedge\ inc(A, q)\ \wedge\ inc(B, p)\ \wedge\ inc(B, q)\ \wedge\ A \neq B\ \Rightarrow\ p = q)$

Axiom N11: $\forall \alpha : plane\ \forall \beta : plane\ \forall A : point\ \forall B : point\ \forall C : point\ (inc(A, \alpha)\ \wedge\ inc(A, \beta)\ \wedge\ inc(B, \alpha)\ \wedge\ inc(B, \beta)\ \wedge\ inc(C, \alpha)\ \wedge\ inc(C, \beta)\ \wedge\ \neg col(A, B, C)\ \Rightarrow\ \alpha = \beta)$

Axiom N12: $\forall \alpha : plane\ \forall p : line\ \forall A : point\ \forall B : point\ (inc(A, \alpha)\ \wedge\ inc(A, p)\ \wedge\ inc(B, \alpha)\ \wedge\ inc(B, p)\ \wedge\ A \neq B\ \Rightarrow\ inc(p, \alpha))$

Axiom G1: $\forall A : point\ \forall B : point\ \forall C : point\ (A \neq B\ \wedge\ A \neq C\ \wedge\ B \neq C\ \wedge\ col(A, B, C)\ \Rightarrow\ bet(A, B, C)\ \vee\ bet(B, C, A)\ \vee\ bet(C, A, B))$

Axiom G2: $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ \forall \alpha : plane\ \forall p : line\ (\neg col(A, B, C)\ \wedge\ inc(A, \alpha)\ \wedge\ inc(B, \alpha)\ \wedge\ inc(C, \alpha)\ \wedge\ inc(p, \alpha)\ \wedge\ \neg inc(A, p)\ \wedge\ inc(D, p)\ \wedge\ bet(B, D, C)\ \Rightarrow\ (\exists E : point\ ((inc(E, p)\ \wedge\ bet(C, E, A)))\ \vee\ (\exists F : point\ (inc(F, p)\ \wedge\ bet(A, F, B)))))$

Axiom P1: $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ \forall E : point\ \forall \alpha : plane\ (\neg col(A, B, C)\ \wedge\ inc(D, \alpha)\ \wedge\ inc(E, \alpha)\ \wedge\ cong(A, B, D, E)\ \Rightarrow\ \exists F : point\ \exists G : point\ (inc(F, \alpha)\ \wedge\ inc(G, \alpha)\ \wedge\ cong(A, C, D, F)\ \wedge\ cong(A, C, D, G)\ \wedge\ cong(B, C, E, F)\ \wedge\ cong(B, C, E, G)))$

Axiom P2: $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ \forall \alpha : plane\ \forall l : line\ (inc(A, \alpha)\ \wedge\ inc(B, \alpha)\ \wedge\ inc(C, \alpha)\ \wedge\ inc(D, \alpha)\ \wedge\ A \neq B\ \wedge\ inc(A, l)\ \wedge\ inc(B, l)\ \wedge\ C \neq D\ \wedge\ \neg inc(C, l)\ \wedge\ \neg inc(D, l)\ \wedge\ cong(A, C, A, D)\ \wedge\ cong(B, C, B, D)\ \Rightarrow\ \exists E : point\ (inc(E, l)\ \wedge\ bet(C, E, D)))$

Axiom P3: $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ \forall E : point\ \forall F : point\ \forall G : point\ \forall H : point\ \forall p : line\ \forall q : line\ (\neg col(A, B, C) \wedge \neg col(D, E, F) \wedge inc(A, p) \wedge inc(B, p) \wedge inc(D, q) \wedge inc(E, q) \wedge inc(G, p) \wedge inc(H, q) \wedge cong(A, B, D, E) \wedge cong(B, C, E, F) \wedge cong(C, A, F, D) \wedge cong(B, G, E, H) \Rightarrow (cong(A, G, D, H)))$

Axiom P4: $\forall \alpha : plane\ \forall \beta : plane\ \forall A : point\ (\alpha \neq \beta \wedge inc(A, \alpha) \wedge inc(A, \beta) \Rightarrow \exists B : point\ (A \neq B \wedge inc(B, \alpha) \wedge inc(B, \beta)))$

Axiom P5: $\forall A : point\ \forall B : point\ (A \neq B \Rightarrow \exists p : line\ (inc(A, p) \wedge inc(B, p)))$

Axiom P6: $\forall A : point\ \forall B : point\ \forall C : point\ (\neg col(A, B, C) \Rightarrow \exists \alpha : plane\ (inc(A, \alpha) \wedge inc(B, \alpha) \wedge inc(C, \alpha)))$

Axiom P7: $\forall p : line\ \exists A : point\ \exists B : point\ (inc(A, p) \wedge A \neq B \wedge inc(B, p))$

Axiom P8: $\forall \alpha : plane\ \exists A : point\ \exists B : point\ \exists C : point\ (inc(A, \alpha) \wedge inc(B, \alpha) \wedge inc(C, \alpha) \wedge \neg col(A, B, C))$

Axiom P9: $\forall A : point\ \forall B : point\ (A \neq B \Rightarrow \exists C : point\ (bet(A, B, C)))$

Axiom P10: $\forall A : point\ \forall B : point\ \forall C : point\ \forall l : line\ (A \neq B \wedge inc(C, l) \Rightarrow \exists D : point\ \exists E : point\ (inc(D, l) \wedge inc(E, l) \wedge cong(A, B, C, D) \wedge cong(A, B, C, E) \wedge bet(D, C, E)))$

Axiom SP1: $\exists A : point\ \exists B : point\ \exists C : point\ \exists D : point\ (\neg comp(A, B, C, D))$

A.5 Tarski's axiom system

Definition Col_1: $\forall A : point\ \forall B : point\ \forall C : point\ (col(A,B,C) \Rightarrow bet(A,B,C) \lor bet(B,C,A) \lor bet(C,A,B))$

Definition Col_2: $\forall A : point\ \forall B : point\ \forall C : point\ (bet(A,B,C) \Rightarrow col(A,B,C))$

Definition Col_3: $\forall A : point\ \forall B : point\ \forall C : point\ (bet(B,C,A) \Rightarrow col(A,B,C))$

Definition Col_4: $\forall A : point\ \forall B : point\ \forall C : point\ (bet(C,A,B) \Rightarrow col(A,B,C))$

Axiom Bet_ident: $\forall A : point\ \forall B : point\ (bet(A,B,A) \Rightarrow A = B)$

Axiom Cong_sym: $\forall A : point\ \forall B : point \Rightarrow cong(A,B,B,A)$

Axiom Cong_ident: $\forall A : point\ \forall B : point\ \forall C : point\ (cong(A,B,C,C) \Rightarrow A = B)$

Axiom Cong_pseudotr: $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ \forall E : point\ \forall F : point\ (cong(A,B,C,D) \land cong(A,B,E,F) \Rightarrow cong(C,D,E,F))$

Axiom Pasch: $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ \forall E : point\ (bet(A,D,C) \land bet(B,E,C) \Rightarrow \exists F : point\ (bet(D,F,B) \land bet(E,F,A)))$

Axiom Euclid: $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ \forall E : point\ (bet(A,B,C) \land bet(D,B,E) \land A \neq B \Rightarrow \exists F : point\ \exists G : point\ (bet(A,D,F) \land bet(A,E,G) \land bet(F,C,G)))$

Axiom Five_segments: $\forall A : point\ \forall A_1 : point\ \forall B : point\ \forall B_1 : point\ \forall C : point\ \forall C_1 : point\ \forall D : point\ \forall D_1 : point\ (cong(A,B,A_1,B_1) \land cong(B,C,B_1,C_1) \land cong(A,D,A_1,D_1) \land cong(B,D,B_1,D_1) \land bet(A,B,C) \land bet(A_1,B_1,C_1) \land A \neq B \Rightarrow cong(C,D,C_1,D_1))$

Axiom Segment_constr: $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point \Rightarrow \exists E : point\ (bet(A,B,E) \land cong(B,E,C,D))$

Axiom Lower_dim: $\Rightarrow \exists A : point\ \exists B : point\ \exists C : point\ (\neg bet(A,B,C) \land \neg bet(B,C,A) \land \neg bet(C,A,B))$

Axiom Upper_dim_3: $\forall A : point\ \forall B : point\ \forall C : point\ \forall D : point\ \forall E : point\ \forall F : point\ (cong(A,B,A,C) \land cong(D,B,D,C) \land cong(E,B,E,C) \land cong(A,B,A,F) \land cong(D,B,D,F) \land cong(E,B,E,F) \land B \neq C \land B \neq$

$F \;\wedge\; C \neq F \;\Rightarrow\; bet(A, D, E) \;\vee\; bet(D, E, A) \;\vee\; bet(E, A, D))$

A.6 List of conjectures

Theorem TH_1:
$\forall p : line \;\forall q : line \;(int(p, q) \;\Rightarrow\; \exists \alpha : plane \;(inc(p, \alpha) \;\wedge\; inc(q, \alpha)))$
Theorem TH_2:
$\forall p : line \;\forall q : line \;\forall A : point \;\forall B : point \;(p \neq q \;\wedge\; inc(A, p) \;\wedge\; inc(A, q) \;\wedge\; inc(B, p) \;\wedge\; inc(B, q) \;\Rightarrow\; A = B)$
Theorem TH_3:
$\forall p : line \;\forall \alpha : plane \;\forall A : point \;\forall B : point \;(\neg inc(p, \alpha) \;\wedge\; inc(A, p) \;\wedge\; inc(A, \alpha) \;\wedge\; inc(B, p) \;\wedge\; inc(B, \alpha) \;\Rightarrow\; A = B)$
Theorem TH_4:
$\forall A : point \;\forall B : point \;(A \neq B \;\Rightarrow\; \exists p : line \;(inc(A, p) \;\wedge\; inc(B, p)))$
Theorem TH_5:
$\forall A : point \;\forall B : point \;\forall p : line \;\forall q : line \;(A \neq B \;\wedge\; inc(A, p) \;\wedge\; inc(B, p) \;\wedge\; inc(A, q) \;\wedge\; inc(B, q) \;\Rightarrow\; p = q)$
Theorem TH_6:
$\forall A : point \;\forall B : point \;\forall C : point \;(\neg col(A, B, C) \;\Rightarrow\; \exists \alpha : plane \;(inc(A, \alpha) \;\wedge\; inc(B, \alpha) \;\wedge\; inc(C, \alpha)))$
Theorem TH_7:
$A : point \;\forall B : point \;\forall C : point \;\forall \alpha : plane \;\forall \beta : plane \;(\neg col(A, B, C) \;\wedge\; inc(A, \alpha) \;\wedge\; inc(B, \alpha) \;\wedge\; inc(C, \alpha) \;\wedge\; inc(A, \beta) \;\wedge\; inc(B, \beta) \;\wedge\; inc(C, \beta) \;\Rightarrow\; \alpha = \beta)$
Theorem TH_8:
$\forall A : point \;\forall p : line \;(\neg inc(A, p) \;\Rightarrow\; \exists \alpha : plane \;(inc(A, \alpha) \;\wedge\; inc(p, \alpha)))$
Theorem TH_9:
$\forall A : point \;\forall B : point \;\forall C : point \;\forall D : point \;\forall \alpha : plane \;(comp(A, B, C, D) \;\wedge\; \neg col(A, B, C) \;\wedge\; inc(A, \alpha) \;\wedge\; inc(B, \alpha) \;\wedge\; inc(C, \alpha) \;\Rightarrow\; inc(D, \alpha))$
Theorem TH_10:
$\forall p : line \;\forall q : line \;\forall r : line \;\forall A : point \;\forall \alpha : plane \;(p \neq q \;\wedge\; q \neq r \;\wedge\; inc(p, \alpha) \;\wedge\; inc(q, \alpha) \;\wedge\; inc(r, \alpha) \;\wedge\; \neg int(p, q) \;\wedge\; \neg int(q, r) \;\wedge\; inc(A, \alpha) \;\wedge\; inc(A, p) \;\wedge\; inc(A, r) \;\Rightarrow\; p = r)$
Theorem TH_11:
$\forall A : point \;\forall B : point \;\forall C : point \;(col(A, B, C) \;\Rightarrow\; col(A, C, B))$
Theorem TH_12:
$\forall A : point \;\forall B : point \;\forall C : point \;(bet(A, B, C) \;\Rightarrow\; bet(C, B, A))$
Theorem TH_13:
$\forall \;A : point \;\forall B : point \;\forall C : point \;(bet(A, B, C) \;\Rightarrow\; \neg bet(A, C, B))$
Theorem TH_14:
$\forall A : point \;\forall B : point \;(A \neq B \;\Rightarrow\; \exists C : point \;(bet(A, C, B)))$
Theorem TH_15:
$\forall A : point \;\forall B : point \;\Rightarrow\; cong(A, B, A, B)$

Theorem TH_16:
$\forall A : point \quad \forall B : point \quad \forall C : point \quad \forall D : point \quad (cong(A, B, C, D) \Rightarrow cong(C, D, A, B))$