Foundations of Geometry

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Chapter 1

Theory description

1.1 Name

Euclidean_geometry

1.2 Signature

Types:

- *point*

1.3 Axioms

Axiom 1 (ax_1) *It holds that $AB \cong BA$.*

Axiom 2 (ax_2) *Assuming that $AB \cong CD$ and $AB \cong EF$ it holds that $CD \cong EF$.*

Axiom 3 (ax_3) *Assuming that $AB \cong CC$ it holds that $A = B$.*

Axiom 4 (ax_4) *There exist point E, such that $bet(D, A, E)$ and $AE \cong BC$.*

Axiom 5 (ax_5) *Assuming that $A \neq B$ and $bet(A, B, C)$ and $bet(E, F, G)$ and $AB \cong EF$ and $BC \cong FG$ and $AD \cong EI$ and $BD \cong FI$ it holds that $CD \cong GI$.*

Axiom 6 (ax_6) *Assuming that $bet(A, B, A)$ it holds that $A = B$.*

Axiom 7 (ax_7) *Assuming that $bet(A, D, C)$ and $bet(B, E, C)$ there exist point F, such that $bet(D, F, B)$ and $bet(E, F, A)$.*

Axiom 8 (ax_branch_bet) *It holds that $bet(A, B, C)$ or $not\_bet(A, B, C)$.*

Axiom 9 (ax_false_bet) *Assuming that $bet(A, B, C)$ and $not\_bet(A, B, C)$ we have contradiction.*

Axiom 10 (ax_8) *There exist point A, point B, point C, such that $not\_bet(A, B, C)$ and $not\_bet(B, C, A)$ and $not\_bet(C, A, B)$.*

Axiom 11 (ax_9) *Assuming that $A \neq B$ and $CA \cong CB$ and $DA \cong DB$ and $EA \cong EB$ it holds that $bet(C, D, E)$ or $bet(D, E, C)$ or $bet(E, C, D)$.*

Axiom 12 (ax_10) *Assuming that $bet(A, D, E)$ and $bet(B, D, C)$ and $A \neq D$ there exist point F, point G, such that $bet(A, B, F)$ and $bet(A, C, G)$ and $bet(F, E, G)$.*

Axiom 13 (ax_g1) *It holds that $A = B$ or $A \neq B$.*

Axiom 14 ($ax\_2\_10\_1$) *Assuming that $afs(A, B, C, D, E, F, G, I)$ it holds that $bet(A, B, C)$ and $bet(E, F, G)$ and $AB \cong EF$ and $BC \cong FG$ and $AD \cong EI$ and $BD \cong FI$.*

Axiom 15 ($ax\_2\_10\_2$) *Assuming that $bet(A, B, C)$ and $bet(E, F, G)$ and $AB \cong EF$ and $BC \cong FG$ and $AD \cong EI$ and $BD \cong FI$ it holds that $afs(A, B, C, D, E, F, G, I)$.*

Axiom 16 (ax_branch_afs) *It holds that $afs(A, B, C, D, E, F, G, I)$ or $not\_afs(A, B, C, D, E, F, G, I)$.*

Axiom 17 (ax_false_afs) *Assuming that $afs(A, B, C, D, E, F, G, I)$ and $not\_afs(A, B, C, D, E, F, G, I)$ we have contradiction.*

Axiom 18 ($ax\_3\_8\_1$) *Assuming that $bet4(A, B, C, D)$ it holds that $bet(A, B, C)$ and $bet(A, B, D)$ and $bet(A, C, D)$ and $bet(B, C, D)$.*

Axiom 19 ($ax\_3\_8\_2$) *Assuming that $bet(A, B, C)$ and $bet(A, B, D)$ and $bet(A, C, D)$ and $bet(B, C, D)$ it holds that $bet4(A, B, C, D)$.*

Axiom 20 (ax_branch_bet4) *It holds that $bet4(A, B, C, D)$ or $not\_bet4(A, B, C, D)$.*

Axiom 21 (ax_false_bet4) *Assuming that $bet4(A, B, C, D)$ and $not\_bet4(A, B, C, D)$ we have contradiction.*

Axiom 22 ($ax\_4\_1\_1$) *Assuming that $ifs(A, B, C, D, E, F, G, I)$ it holds that $bet(A, B, C)$ and $bet(E, F, G)$ and $AC \cong EG$ and $BC \cong FG$ and $AD \cong EI$ and $CD \cong GI$.*

Axiom 23 ($ax\_4\_1\_2$) *Assuming that $bet(A, B, C)$ and $bet(E, F, G)$ and $AC \cong EG$ and $BC \cong FG$ and $AD \cong EI$ and $CD \cong GI$ it holds that $ifs(A, B, C, D, E, F, G, I)$.*

Axiom 24 (ax_branch_ifs) *It holds that $ifs(A, B, C, D, E, F, G, I)$ or $not\_ifs(A, B, C, D, E, F, G, I)$.*

Axiom 25 (ax_false_ifs) *Assuming that $ifs(A, B, C, D, E, F, G, I)$ and $not\_ifs(A, B, C, D, E, F, G, I)$ we have contradiction.*

Axiom 26 $\left(ax\_4\_4\_1\right)$ *Assuming that* $cong3(A, B, C, D, E, F)$ *it holds that* $AB \cong DE$ *and* $AC \cong DF$ *and* $BC \cong EF$.

Axiom 27 $\left(ax\_4\_4\_2\right)$ *Assuming that* $AB \cong DE$ *and* $AC \cong DF$ *and* $BC \cong EF$ *it holds that* $cong3(A, B, C, D, E, F)$.

Axiom 28 $\left(ax\_4\_4\_3\right)$ *Assuming that* $cong4(A, B, C, D, E, F, G, I)$ *it holds that* $AB \cong EF$ *and* $AC \cong EG$ *and* $AD \cong EI$ *and* $BC \cong FG$ *and* $BD \cong FI$ *and* $CD \cong GI$.

Axiom 29 $\left(ax\_4\_4\_4\right)$ *Assuming that* $AB \cong EF$ *and* $AC \cong EG$ *and* $AD \cong EI$ *and* $BC \cong FG$ *and* $BD \cong FI$ *and* $CD \cong GI$ *it holds that* $cong4(A, B, C, D, E, F, G, I)$.

Axiom 30 $\left(ax\_branch\_cong3\right)$ *It holds that* $cong3(A, B, C, D, E, F)$ *or* $not\_cong3(A, B, C, D, E, F)$.

Axiom 31 $\left(ax\_false\_cong3\right)$ *Assuming that* $cong3(A, B, C, D, E, F)$ *and* $not\_cong3(A, B, C, D, E, F)$ *we have contradiction.*

Axiom 32 $\left(ax\_branch\_cong4\right)$ *It holds that* $cong4(A, B, C, D, E, F, G, I)$ *or* $not\_cong4(A, B, C, D, E, F, G, I)$.

Axiom 33 $\left(ax\_false\_cong4\right)$ *Assuming that* $cong4(A, B, C, D, E, F, G, I)$ *and* $not\_cong4(A, B, C, D, E, F, G, I)$ *we have contradiction.*

Axiom 34 $\left(ax\_4\_10\_1\right)$ *Assuming that* $col(A, B, C)$ *it holds that* $bet(A, B, C)$ *or* $bet(B, C, A)$ *or* $bet(C, A, B)$.

Axiom 35 $\left(ax\_4\_10\_2\right)$ *Assuming that* $bet(A, B, C)$ *it holds that* $col(A, B, C)$.

Axiom 36 $\left(ax\_4\_10\_3\right)$ *Assuming that* $bet(B, C, A)$ *it holds that* $col(A, B, C)$.

Axiom 37 $\left(ax\_4\_10\_4\right)$ *Assuming that* $bet(C, A, B)$ *it holds that* $col(A, B, C)$.

Axiom 38 $\left(ax\_4\_15\_1\right)$ *Assuming that* $fs(A, B, C, D, E, F, G, I)$ *it holds that* $col(A, B, C)$ *and* $cong3(A, B, C, E, F, G)$ *and* $AD \cong EI$ *and* $BD \cong FI$.

Axiom 39 $\left(ax\_4\_15\_2\right)$ *Assuming that* $col(A, B, C)$ *and* $cong3(A, B, C, E, F, G)$ *and* $AD \cong EI$ *and* $BD \cong FI$ *it holds that* $fs(A, B, C, D, E, F, G, I)$.

Axiom 40 $\left(ax\_branch\_fs\right)$ *It holds that* $fs(A, B, C, D, E, F, G, I)$ *or* $not\_fs(A, B, C, D, E, F, G, I)$.

Axiom 41 $\left(ax\_false\_fs\right)$ *Assuming that* $fs(A, B, C, D, E, F, G, I)$ *and* $not\_fs(A, B, C, D, E, F, G, I)$ *we have contradiction.*

Axiom 42 $\left(ax\_5\_4\_1\right)$ *Assuming that* $le(A, B, C, D)$ *there exist point E, such that* $bet(C, E, D)$ *and* $AB \cong CE$.

Axiom 43 $\left(ax\_5\_4\_2\right)$ *Assuming that* $bet(C, E, D)$ *and* $AB \cong CE$ *it holds that* $le(A, B, C, D)$.

Axiom 44 $\left(ax\_5\_4\_3\right)$ *Assuming that* $ge(C, D, A, B)$ *it holds that* $le(A, B, C, D)$.

Axiom 45 ($ax\_5\_4\_4$) *Assuming that* $le(A, B, C, D)$ *it holds that* $ge(C, D, A, B)$.

Axiom 46 (ax_branch_le) *It holds that* $le(A, B, C, D)$ *or* $not\_le(A, B, C, D)$.

Axiom 47 (ax_false_le) *Assuming that* $le(A, B, C, D)$ *and* $not\_le(A, B, C, D)$ *we have contradiction.*

Axiom 48 (ax_branch_ge) *It holds that* $ge(A, B, C, D)$ *or* $not\_ge(A, B, C, D)$.

Axiom 49 (ax_false_ge) *Assuming that* $ge(A, B, C, D)$ *and* $not\_ge(A, B, C, D)$ *we have contradiction.*

Axiom 50 (ax_branch_cong) *It holds that* $AB \cong CD$ *or* $not\_cong(A, B, C, D)$.

Axiom 51 (ax_false_cong) *Assuming that* $AB \cong CD$ *and* $not\_cong(A, B, C, D)$ *we have contradiction.*

Axiom 52 ($ax\_5\_14\_1$) *Assuming that* $lt(A, B, C, D)$ *it holds that* $le(A, B, C, D)$ *and* $not\_cong(A, B, C, D)$.

Axiom 53 ($ax\_5\_14\_2$) *Assuming that* $le(A, B, C, D)$ *and* $not\_cong(A, B, C, D)$ *it holds that* $lt(A, B, C, D)$.

Axiom 54 ($ax\_5\_14\_3$) *Assuming that* $gt(C, D, A, B)$ *it holds that* $lt(A, B, C, D)$.

Axiom 55 ($ax\_5\_14\_4$) *Assuming that* $lt(A, B, C, D)$ *it holds that* $gt(C, D, A, B)$.

Axiom 56 (ax_branch_lt) *It holds that* $lt(A, B, C, D)$ *or* $not\_lt(A, B, C, D)$.

Axiom 57 (ax_false_lt) *Assuming that* $lt(A, B, C, D)$ *and* $not\_lt(A, B, C, D)$ *we have contradiction.*

Axiom 58 (ax_branch_gt) *It holds that* $gt(A, B, C, D)$ *or* $not\_gt(A, B, C, D)$.

Axiom 59 (ax_false_gt) *Assuming that* $gt(A, B, C, D)$ *and* $not\_gt(A, B, C, D)$ *we have contradiction.*

Axiom 60 ($ax\_6\_1\_1$) *Assuming that* $out(C, A, B)$ *it holds that* $A \neq C$ *and* $B \neq C$ *and* $bet(C, A, B)$ *or* $A \neq C$ *and* $B \neq C$ *and* $bet(C, B, A)$.

Axiom 61 ($ax\_6\_1\_2$) *Assuming that* $A \neq C$ *and* $B \neq C$ *and* $bet(C, A, B)$ *it holds that* $out(C, A, B)$.

Axiom 62 ($ax\_6\_1\_3$) *Assuming that* $A \neq C$ *and* $B \neq C$ *and* $bet(C, B, A)$ *it holds that* $out(C, A, B)$.

Axiom 63 (ax_branch_out) *It holds that* $out(C, A, B)$ *or* $not\_out(C, A, B)$.

Axiom 64 (ax_false_out) *Assuming that* $out(C, A, B)$ *and* $not\_out(C, A, B)$ *we have contradiction.*

Axiom 65 ($ax\_6\_8\_1$) *Assuming that* $B \neq A$ *and* $point\_on\_ray(C, A, B)$ *it holds that* $out(A, C, B)$.

Axiom 66 ($ax\_6\_8\_2$) *Assuming that* $B \neq A$ *and* $out(A, C, B)$ *it holds that* $point\_on\_ray(C, A, B)$.

Axiom 67 ($ax\_branch\_point\_on\_ray$) *It holds that* $point\_on\_ray(C, A, B)$ *or* $not\_point\_on\_ray(C, A, B)$.

Axiom 68 ($ax\_false\_point\_on\_ray$) *Assuming that* $point\_on\_ray(C, A, B)$ *and* $not\_point\_on\_ray(C, A, B)$ *we have contradiction.*

Axiom 69 ($ax\_6\_14\_1$) *Assuming that* $point\_on\_line(C, A, B)$ *it holds that* $A \neq B$ *and* $col(A, B, C)$.

Axiom 70 ($ax\_6\_14\_2$) *Assuming that* $A \neq B$ *and* $col(A, B, C)$ *it holds that* $point\_on\_line(C, A, B)$.

Axiom 71 ($ax\_same\_lines\_1$) *Assuming that* $A \neq B$ *and* $C \neq D$ *and* $point\_on\_line(C, A, B)$ *and* $point\_on\_line(D, A, B)$ *it holds that* $same\_lines(A, B, C, D)$.

Axiom 72 ($ax\_same\_lines\_2$) *Assuming that* $same\_lines(A, B, C, D)$ *it holds that* $A \neq B$ *and* $C \neq D$ *and* $point\_on\_line(C, A, B)$ *and* $point\_on\_line(D, A, B)$.

Axiom 73 ($ax\_false\_same\_lines$) *Assuming that* $same\_lines(A, B, C, D)$ *and* $not\_same\_lines(A, B, C, D)$ *we have contradiction.*

Axiom 74 ($ax\_branch\_same\_lines$) *It holds that* $same\_lines(A, B, C, D)$ *or* $not\_same\_lines(A, B, C, D)$.

Axiom 75 (ax_branch_col) *It holds that* $col(A, B, C)$ *or* $not\_col(A, B, C)$.

Axiom 76 (ax_false_col) *Assuming that* $col(A, B, C)$ *and* $not\_col(A, B, C)$ *we have contradiction.*

Axiom 77 ($ax\_branch\_point\_on\_line$) *Assuming that* $A \neq B$ *it holds that* $point\_on\_line(C, A, B)$ *or* $not\_point\_on\_line(C, A, B)$.

Axiom 78 ($ax\_false\_point\_on\_line$) *Assuming that* $point\_on\_line(C, A, B)$ *and* $not\_point\_on\_line(C, A, B)$ *we have contradiction.*

Axiom 79 ($ax\_6\_22\_1$) *Assuming that* $inter(E, A, B, C, D)$ *it holds that* $A \neq B$ *and* $C \neq D$ *and* $point\_on\_line(E, A, B)$ *and* $point\_on\_line(E, C, D)$ *and* $not\_same\_lines(A, B, C, D)$.

Axiom 80 ($ax\_6\_22\_2$) *Assuming that* $A \neq B$ *and* $C \neq D$ *and* $point\_on\_line(E, A, B)$ *and* $point\_on\_line(E, C, D)$ *and* $not\_same\_lines(A, B, C, D)$ *it holds that* $inter(E, A, B, C, D)$.

Axiom 81 (ax_branch_inter) *It holds that* $inter(A, B, C, D, E)$ *or* $not\_inter(A, B, C, D, E)$.

Axiom 82 (ax_false_inter) *Assuming that* $inter(A, B, C, D, E)$ *and* $not\_inter(A, B, C, D, E)$ *we have contradiction.*

Axiom 83 (ax_7_1) *Assuming that $is\_midpoint(B, A, C)$ it holds that $bet(A, B, C)$ and $BA \cong BC$.*

Axiom 84 (ax_7_2) *Assuming that $bet(A, B, C)$ and $BA \cong BC$ it holds that $is\_midpoint(B, A, C)$.*

Axiom 85 ($ax\_branch\_is\_midpoint$) *It holds that $is\_midpoint(A, B, C)$ or $not\_is\_midpoint(A, B, C)$.*

Axiom 86 ($ax\_false\_is\_midpoint$) *Assuming that $is\_midpoint(A, B, C)$ and $not\_is\_midpoint(A, B, C)$ we have contradiction.*

Axiom 87 ($ax\_7\_5\_1$) *Assuming that $is\_symmetric(A, B, C)$ it holds that $is\_midpoint(C, A, B)$.*

Axiom 88 ($ax\_7\_5\_2$) *Assuming that $is\_midpoint(C, A, B)$ it holds that $is\_symmetric(A, B, C)$.*

Axiom 89 ($ax\_branch\_is\_symmetric$) *It holds that $is\_symmetric(A, B, C)$ or $not\_is\_symmetric(A, B, C)$.*

Axiom 90 ($ax\_false\_is\_symmetric$) *Assuming that $is\_symmetric(A, B, C)$ and $not\_is\_symmetric(A, B, C)$ we have contradiction.*

Axiom 91 ($ax\_8\_1\_1$) *Assuming that $per(A, B, C)$ there exist point D, such that $AC \cong AD$ and $is\_midpoint(B, C, D)$.*

Axiom 92 ($ax\_8\_1\_2$) *Assuming that $AC \cong AD$ and $is\_midpoint(B, C, D)$ it holds that $per(A, B, C)$.*

Axiom 93 (ax_branch_per) *It holds that $per(A, B, C)$ or $not\_per(A, B, C)$.*

Axiom 94 (ax_false_per) *Assuming that $per(A, B, C)$ and $not\_per(A, B, C)$ we have contradiction.*

Axiom 95 ($ax\_8\_11\_1\_1$) *Assuming that $perp\_in(A, B, C, D, E)$ and $B \neq A$ and $D \neq A$ it holds that $B \neq C$ and $D \neq E$ and $point\_on\_line(A, B, C)$ and $point\_on\_line(A, D, E)$ and $per(B, A, D)$.*

Axiom 96 ($ax\_8\_11\_1\_2$) *Assuming that $perp\_in(A, B, C, D, E)$ and $B \neq A$ and $E \neq A$ it holds that $B \neq C$ and $D \neq E$ and $point\_on\_line(A, B, C)$ and $point\_on\_line(A, D, E)$ and $per(B, A, E)$.*

Axiom 97 ($ax\_8\_11\_1\_3$) *Assuming that $perp\_in(A, B, C, D, E)$ and $C \neq A$ and $D \neq A$ it holds that $B \neq C$ and $D \neq E$ and $point\_on\_line(A, B, C)$ and $point\_on\_line(A, D, E)$ and $per(C, A, D)$.*

Axiom 98 ($ax\_8\_11\_1\_4$) *Assuming that $perp\_in(A, B, C, D, E)$ and $C \neq A$ and $E \neq A$ it holds that $B \neq C$ and $D \neq E$ and $point\_on\_line(A, B, C)$ and $point\_on\_line(A, D, E)$ and $per(C, A, E)$.*

Axiom 99 (*ax_8_11_1_5*) *Assuming that $B \neq A$ and $D \neq A$ and $B \neq C$ and $D \neq E$ and $point\_on\_line(A, B, C)$ and $point\_on\_line(A, D, E)$ and $per(B, A, D)$ it holds that $perp\_in(A, B, C, D, E)$.*

Axiom 100 (*ax_8_11_1_6*) *Assuming that $B \neq A$ and $E \neq A$ and $B \neq C$ and $D \neq E$ and $point\_on\_line(A, B, C)$ and $point\_on\_line(A, D, E)$ and $per(B, A, E)$ it holds that $perp\_in(A, B, C, D, E)$.*

Axiom 101 (*ax_8_11_1_7*) *Assuming that $C \neq A$ and $D \neq A$ and $B \neq C$ and $D \neq E$ and $point\_on\_line(A, B, C)$ and $point\_on\_line(A, D, E)$ and $per(C, A, D)$ it holds that $perp\_in(A, B, C, D, E)$.*

Axiom 102 (*ax_8_11_1_8*) *Assuming that $C \neq A$ and $E \neq A$ and $B \neq C$ and $D \neq E$ and $point\_on\_line(A, B, C)$ and $point\_on\_line(A, D, E)$ and $per(C, A, E)$ it holds that $perp\_in(A, B, C, D, E)$.*

Axiom 103 (*ax_8_11_2_1*) *Assuming that $perp(A, B, C, D)$ there exist point E, such that $perp\_in(E, A, B, C, D)$.*

Axiom 104 (*ax_8_11_2_2*) *Assuming that $perp\_in(A, B, C, D, E)$ it holds that $perp(B, C, D, E)$.*

Axiom 105 (*ax_branch_perp_in*) *It holds that $perp\_in(A, B, C, D, E)$ or $not\_perp\_in(A, B, C, D, E)$.*

Axiom 106 (*ax_false_perp_in*) *Assuming that $perp\_in(A, B, C, D, E)$ and $not\_perp\_in(A, B, C, D, E)$ we have contradiction.*

Axiom 107 (*ax_branch_perp*) *It holds that $perp(A, B, C, D)$ or $not\_perp(A, B, C, D)$.*

Axiom 108 (*ax_false_perp*) *Assuming that $perp(A, B, C, D)$ and $not\_perp(A, B, C, D)$ we have contradiction.*

Axiom 109 (*ax_9_1_1*) *Assuming that $two\_sides(C, D, A, B)$ there exist point E, such that $A \neq B$ and $not\_point\_on\_line(C, A, B)$ and $not\_point\_on\_line(D, A, B)$ and $point\_on\_line(E, A, B)$ and $bet(C, E, D)$.*

Axiom 110 (*ax_9_1_2*) *Assuming that $A \neq B$ and $not\_point\_on\_line(C, A, B)$ and $not\_point\_on\_line(D, A, B)$ and $point\_on\_line(E, A, B)$ and $bet(C, E, D)$ it holds that $two\_sides(C, D, A, B)$.*

Axiom 111 (*ax_9_7_1*) *Assuming that $one\_side(C, D, A, B)$ there exist point E, such that $A \neq B$ and $two\_sides(C, E, A, B)$ and $two\_sides(D, E, A, B)$.*

Axiom 112 (*ax_9_7_2*) *Assuming that $A \neq B$ and $two\_sides(C, E, A, B)$ and $two\_sides(D, E, A, B)$ it holds that $one\_side(C, D, A, B)$.*

Axiom 113 (*ax_branch_two_sides*) *Assuming that $A \neq B$ it holds that $two\_sides(C, D, A, B)$ or $not\_two\_sides(C, D, A, B)$.*

Axiom 114 ($ax\_false\_two\_sides$) *Assuming that $two\_sides(C, D, A, B)$ and $not\_two\_sides(C, D, A, B)$ we have contradiction.*

Axiom 115 ($ax\_branch\_one\_side$) *Assuming that $A \neq B$ it holds that $one\_side(C, D, A, B)$ or $not\_one\_side(C, D, A, B)$.*

Axiom 116 ($ax\_false\_one\_side$) *Assuming that $one\_side(C, D, A, B)$ and $not\_one\_side(C, D, A, B)$ we have contradiction.*

Axiom 117 ($ax\_10\_3\_1$) *Assuming that $C \neq D$ and $is\_image\_spec(B, A, C, D)$ there exist point E, such that $is\_midpoint(E, A, B)$ and $point\_on\_line(E, C, D)$ and $perp(C, D, A, B)$ or $is\_midpoint(E, A, B)$ and $point\_on\_line(E, C, D)$ and $A = B$.*

Axiom 118 ($ax\_10\_3\_2\_1$) *Assuming that $C \neq D$ and $is\_midpoint(E, A, B)$ and $point\_on\_line(E, C, D)$ and $perp(C, D, A, B)$ it holds that $is\_image\_spec(B, A, C, D)$.*

Axiom 119 ($ax\_10\_3\_2\_2$) *Assuming that $B \neq C$ and $is\_midpoint(D, A, A)$ and $point\_on\_line(D, B, C)$ it holds that $is\_image\_spec(A, A, B, C)$.*

Axiom 120 ($ax\_branch\_is\_image\_spec$) *Assuming that $C \neq D$ it holds that $is\_image\_spec(A, B, C, D)$ or $not\_is\_image\_spec(A, B, C, D)$.*

Axiom 121 ($ax\_false\_is\_image\_spec$) *Assuming that $is\_image\_spec(A, B, C, D)$ and $not\_is\_image\_spec(A, B, C, D)$ we have contradiction.*

Axiom 122 ($ax\_10\_3\_3$) *Assuming that $is\_image(B, A, C, D)$ it holds that $C \neq D$ and $is\_image\_spec(B, A, C, D)$ or $C = D$ and $is\_midpoint(C, A, B)$.*

Axiom 123 ($ax\_10\_3\_4\_1$) *Assuming that $C \neq D$ and $is\_image\_spec(B, A, C, D)$ it holds that $is\_image(B, A, C, D)$.*

Axiom 124 ($ax\_10\_3\_4\_2$) *Assuming that $is\_midpoint(C, A, B)$ it holds that $is\_image(B, A, C, C)$.*

Axiom 125 ($ax\_branch\_is\_image$) *It holds that $is\_image(B, A, C, D)$ or $not\_is\_image(B, A, C, D)$.*

Axiom 126 ($ax\_false\_is\_image$) *Assuming that $is\_image(B, A, C, D)$ and $not\_is\_image(B, A, C, D)$ we have contradiction.*

Axiom 127 ($ax\_is\_image\_spec\_in\_1$) *Assuming that $D \neq E$ and $is\_image\_spec\_in(A, B, C, D, E)$ it holds that $is\_midpoint(A, C, B)$ and $col(D, E, A)$ and $perp(D, E, C, B)$ or $is\_midpoint(A, C, B)$ and $col(D, E, A)$ and $C = B$.*

Axiom 128 ($ax\_is\_image\_spec\_in\_2$) *Assuming that $D \neq E$ and $is\_midpoint(A, C, B)$ and $col(D, E, A)$ and $perp(D, E, C, B)$ it holds that $is\_image\_spec\_in(A, B, C, D, E)$.*

Axiom 129 ($ax\_is\_image\_spec\_in\_3$) *Assuming that $C \neq D$ and $is\_midpoint(A, B, B)$ and $col(C, D, A)$ it holds that $is\_image\_spec\_in(A, B, B, C, D)$.*

Axiom 130 ($ax\_is\_image\_spec\_in\_gen\_1$) *Assuming that $is\_image\_spec\_in\_gen(A, B, C, D, E)$ it holds that $D \neq E$ and $is\_image\_spec\_in(A, B, C, D, E)$ or $D = E$ and $D = A$ and $is\_midpoint(A, C, B)$.*

Axiom 131 ($ax\_is\_image\_spec\_in\_gen\_2$) *Assuming that $D \neq E$ and $is\_image\_spec\_in(A, B, C, D, E)$ it holds that $is\_image\_spec\_in\_gen(A, B, C, D, E)$.*

Axiom 132 ($ax\_is\_image\_spec\_in\_gen\_3$) *Assuming that $is\_midpoint(A, C, B)$ it holds that $is\_image\_spec\_in\_gen(A, B, C, A, A)$.*

Axiom 133 ($ax\_false\_is\_image\_spec\_in$) *Assuming that $is\_image\_spec\_in(A, B, C, D, E)$ and $not\_is\_image\_spec\_in(A, B, C, D, E)$ we have contradiction.*

Axiom 134 ($ax\_branch\_is\_image\_spec\_in$) *Assuming that $D \neq E$ it holds that $is\_image\_spec\_in(A, B, C, D, E)$ or $not\_is\_image\_spec\_in(A, B, C, D, E)$.*

Axiom 135 ($ax\_branch\_is\_image\_spec\_in\_gen$) *It holds that $is\_image\_spec\_in\_gen(A, B, C, D, E)$ or $not\_is\_image\_spec\_in\_gen(A, B, C, D, E)$.*

Axiom 136 ($ax\_false\_is\_image\_spec\_in\_gen$) *Assuming that $is\_image\_spec\_in\_gen(A, B, C, D, E)$ and $not\_is\_image\_spec\_in\_gen(A, B, C, D, E)$ we have contradiction.*

Axiom 137 ($ax\_11\_2\_1$) *Assuming that $cong\_angle(A, B, C, D, E, F)$ there exist point G, point I, point J, point K, such that $A \neq B$ and $C \neq B$ and $D \neq E$ and $F \neq E$ and $bet(B, A, G)$ and $AG \cong ED$ and $bet(B, C, I)$ and $CI \cong EF$ and $bet(E, D, J)$ and $DJ \cong BA$ and $bet(E, F, K)$ and $FK \cong BC$ and $GI \cong JK$.*

Axiom 138 ($ax\_11\_2\_2$) *Assuming that $A \neq B$ and $C \neq B$ and $D \neq E$ and $F \neq E$ and $bet(B, A, G)$ and $AG \cong ED$ and $bet(B, C, I)$ and $CI \cong EF$ and $bet(E, D, J)$ and $DJ \cong BA$ and $bet(E, F, K)$ and $FK \cong BC$ and $GI \cong JK$ it holds that $cong\_angle(A, B, C, D, E, F)$.*

Axiom 139 ($ax\_distinct\_1$) *Assuming that $distinct(A, B, C)$ it holds that $A \neq B$ and $A \neq C$ and $B \neq C$.*

Axiom 140 ($ax\_distinct\_2$) *Assuming that $A \neq B$ and $A \neq C$ and $B \neq C$ it holds that $distinct(A, B, C)$.*

Axiom 141 ($ax\_branch\_distinct$) *It holds that $distinct(A, B, C)$ or $not\_distinct(A, B, C)$.*

Axiom 142 ($ax\_false\_distinct$) *Assuming that $distinct(A, B, C)$ and $not\_distinct(A, B, C)$ we have contradiction.*

Axiom 143 ($ax\_11\_23\_1\_1$) *Assuming that $point\_in\_angle(D, A, B, C)$ there exist point E, such that $A \neq B$ and $C \neq B$ and $D \neq B$ and $bet(A, E, C)$ and $E = B$ or $A \neq B$ and $C \neq B$ and $D \neq B$ and $bet(A, E, C)$ and $out(B, E, D)$.*

Axiom 144 ($ax\_11\_23\_1\_2$) *Assuming that $A \neq B$ and $C \neq B$ and $D \neq B$ and $bet(A, B, C)$ it holds that $point\_in\_angle(D, A, B, C)$.*

Axiom 145 (*ax_11_23_2*) *Assuming that $A \neq B$ and $C \neq B$ and $D \neq B$ and*
bet(A, E, C) and out(B, E, D) it holds that point_in_angle(D, A, B, C).

Axiom 146 (*ax_branch_point_in_angle*) *It holds that point_in_angle(A, B, C, D)*
or not_point_in_angle(A, B, C, D).

Axiom 147 (*ax_false_point_in_angle*) *Assuming that point_in_angle(A, B, C, D)*
and not_point_in_angle(A, B, C, D) we have contradiction.

Axiom 148 (*ax_11_27_1*) *Assuming that le_angle(A, B, C, D, E, F) there exist*
point G, such that point_in_angle(G, D, E, F) and cong_angle(A, B, C, D, E, G).

Axiom 149 (*ax_11_27_2*) *Assuming that point_in_angle(G, D, E, F) and cong_angle(A, B, C, D, E, G)*
it holds that le_angle(A, B, C, D, E, F).

Axiom 150 (*ax_branch_le_angle*) *It holds that le_angle(A, B, C, D, E, F) or*
not_le_angle(A, B, C, D, E, F).

Axiom 151 (*ax_false_le_angle*) *Assuming that le_angle(A, B, C, D, E, F) and*
not_le_angle(A, B, C, D, E, F) we have contradiction.

Axiom 152 (*ax_11_27_3*) *Assuming that ge_angle(A, B, C, D, E, F) it holds*
that le_angle(D, E, F, A, B, C).

Axiom 153 (*ax_11_27_4*) *Assuming that le_angle(D, E, F, A, B, C) it holds that*
ge_angle(A, B, C, D, E, F).

Axiom 154 (*ax_branch_ge_angle*) *It holds that ge_angle(A, B, C, D, E, F) or*
not_le_angle(A, B, C, D, E, F).

Axiom 155 (*ax_false_ge_angle*) *Assuming that ge_angle(A, B, C, D, E, F) and*
not_ge_angle(A, B, C, D, E, F) we have contradiction.

Axiom 156 (*ax_branch_cong_angle*) *It holds that cong_angle(A, B, C, D, E, F)*
or not_cong_angle(A, B, C, D, E, F).

Axiom 157 (*ax_cong_angle_false*) *Assuming that cong_angle(A, B, C, D, E, F)*
and not_cong_angle(A, B, C, D, E, F) we have contradiction.

Axiom 158 (*ax_11_38_1*) *Assuming that lt_angle(A, B, C, D, E, F) it holds that*
le_angle(A, B, C, D, E, F) and not_cong_angle(A, B, C, D, E, F).

Axiom 159 (*ax_11_38_2*) *Assuming that le_angle(A, B, C, D, E, F) and not_cong_angle(A, B, C, D, E, F)*
it holds that lt_angle(A, B, C, D, E, F).

Axiom 160 (*ax_11_38_3*) *Assuming that gt_angle(A, B, C, D, E, F) it holds that*
lt_angle(D, E, F, A, B, C).

Axiom 161 (*ax_11_38_4*) *Assuming that lt_angle(A, B, C, D, E, F) it holds that*
gt_angle(D, E, F, A, B, C).

Axiom 162 ($ax\_branch\_gt\_angle$) *It holds that* $gt\_angle(A, B, C, D, E, F)$ *or* $not\_gt\_angle(A, B, C, D, E, F)$.

Axiom 163 ($ax\_false\_gt\_angle$) *Assuming that* $gt\_angle(A, B, C, D, E, F)$ *and* $not\_gt\_angle(A, B, C, D, E, F)$ *we have contradiction.*

Axiom 164 ($ax\_branch\_lt\_angle$) *It holds that* $lt\_angle(A, B, C, D, E, F)$ *or* $not\_lt\_angle(A, B, C, D, E, F)$.

Axiom 165 ($ax\_false\_lt\_angle$) *Assuming that* $lt\_angle(A, B, C, D, E, F)$ *and* $not\_lt\_angle(A, B, C, D, E, F)$ *we have contradiction.*

Axiom 166 ($ax\_11\_39\_1$) *Assuming that* $acute(A, B, C)$ *there exist point D, point E, point F, such that* $per(D, E, F)$ *and* $lt\_angle(A, B, C, D, E, F)$.

Axiom 167 ($ax\_11\_39\_2$) *Assuming that* $per(D, E, F)$ *and* $lt\_angle(A, B, C, D, E, F)$ *it holds that* $acute(A, B, C)$.

Axiom 168 ($ax\_11\_39\_3$) *Assuming that* $obtuse(A, B, C)$ *there exist point D, point E, point F, such that* $per(D, E, F)$ *and* $gt\_angle(A, B, C, D, E, F)$.

Axiom 169 ($ax\_11\_39\_4$) *Assuming that* $per(D, E, F)$ *and* $gt\_angle(A, B, C, D, E, F)$ *it holds that* $obtuse(A, B, C)$.

Axiom 170 (ax_branch_acute) *It holds that* $acute(A, B, C)$ *or* $not\_acute(A, B, C)$.

Axiom 171 (ax_false_acute) *Assuming that* $acute(A, B, C)$ *and* $not\_acute(A, B, C)$ *we have contradiction.*

Axiom 172 (ax_branch_obtuse) *It holds that* $obtuse(A, B, C)$ *or* $not\_obtuse(A, B, C)$.

Axiom 173 (ax_false_obtuse) *Assuming that* $obtuse(A, B, C)$ *and* $not\_obtuse(A, B, C)$ *we have contradiction.*

Axiom 174 (ax_int1) *Assuming that* $int(A, B, C, D)$ *there exist point E, such that* $inter(E, A, B, C, D)$.

Axiom 175 (ax_int2) *Assuming that* $inter(E, A, B, C, D)$ *it holds that* $int(A, B, C, D)$.

Axiom 176 (ax_branch_int) *It holds that* $int(A, B, C, D)$ *or* $not\_int(A, B, C, D)$.

Axiom 177 (ax_false_int) *Assuming that* $int(A, B, C, D)$ *and* $not\_int(A, B, C, D)$ *we have contradiction.*

Axiom 178 ($ax\_12\_2\_1$) *Assuming that* $A \neq B$ *and* $C \neq D$ *and* $not\_int(A, B, C, D)$ *it holds that* $parallel(A, B, C, D)$.

Axiom 179 ($ax\_12\_2\_2$) *Assuming that* $parallel(A, B, C, D)$ *it holds that* $A \neq B$ *and* $C \neq D$ *and* $not\_int(A, B, C, D)$.

Axiom 180 ($ax\_12\_3\_1$) *Assuming that parallel_broad(A, B, C, D) it holds that $A \neq B$ and $C \neq D$ and parallel(A, B, C, D) or $A \neq B$ and $C \neq D$ and same_lines(A, B, C, D).*

Axiom 181 ($ax\_12\_3\_2$) *Assuming that $A \neq B$ and $C \neq D$ and parallel(A, B, C, D) it holds that parallel_broad(A, B, C, D).*

Axiom 182 ($ax\_12\_3\_3$) *Assuming that $A \neq B$ and $C \neq D$ and same_lines(A, B, C, D) it holds that parallel_broad(A, B, C, D).*

Axiom 183 ($ax\_branch\_parallel$) *It holds that parallel(A, B, C, D) or not_parallel(A, B, C, D).*

Axiom 184 ($ax\_false\_parallel$) *Assuming that parallel(A, B, C, D) and not_parallel(A, B, C, D) we have contradiction.*

Axiom 185 ($ax\_branch\_parallel\_broad$) *It holds that parallel_broad(A, B, C, D) or not_parallel_broad(A, B, C, D).*

Axiom 186 ($ax\_false\_parallel\_broad$) *Assuming that parallel_broad(A, B, C, D) and not_parallel_broad(A, B, C, D) we have contradiction.*

Chapter 2

Consequences of the Axioms A1-A5

Theorem 1 (th_2_1.) *It holds that $AB \cong AB$.*

   Proof:
1. It holds that $BA \cong AB$ (using ax_1).
2. From the facts $BA \cong AB$ and $BA \cong AB$ it holds that $AB \cong AB$ (using ax_2).
3. From the facts $AB \cong AB$ we proved a conjecture.
QED

Theorem 2 (th_2_2.) *Assuming that $AB \cong CD$ it holds that $CD \cong AB$.*

   Proof:
1. It holds that $BA \cong AB$ (using ax_1).
2. From the facts $BA \cong AB$ and $BA \cong AB$ it holds that $AB \cong AB$ (using ax_2).
3. From the facts $AB \cong CD$ and $AB \cong AB$ it holds that $CD \cong AB$ (using ax_2).
4. From the facts $CD \cong AB$ we proved a conjecture.
QED

Theorem 3 (th_2_3.) *Assuming that $AB \cong CD$ and $CD \cong EF$ it holds that $AB \cong EF$.*

   Proof:
1. From the facts $AB \cong CD$ it holds that $CD \cong AB$ (using th_2_2).
2. From the facts $CD \cong AB$ and $CD \cong EF$ it holds that $AB \cong EF$ (using ax_2).
3. From the facts $AB \cong EF$ we proved a conjecture.

QED

Theorem 4 (th_2_4.) *Assuming that $AB \cong CD$ it holds that $BA \cong CD$.*

   Proof:

1. It holds that $AB \cong BA$ (using ax_1).
2. From the facts $AB \cong BA$ and $AB \cong CD$ it holds that $BA \cong CD$ (using ax_2).
3. From the facts $BA \cong CD$ we proved a conjecture.

QED

Theorem 5 (th_2_5.) *Assuming that $AB \cong CD$ it holds that $AB \cong DC$.*

   Proof:

1. From the facts $AB \cong CD$ it holds that $CD \cong AB$ (using th_2_2).
2. From the facts $CD \cong AB$ it holds that $DC \cong AB$ (using th_2_4).
3. From the facts $DC \cong AB$ it holds that $AB \cong DC$ (using th_2_2).
4. From the facts $AB \cong DC$ we proved a conjecture.

QED

Theorem 6 (th_2_8.) *It holds that $AA \cong BB$.*

   Proof:

1. There exist a point C where $bet(A, A, C)$ and $AC \cong AA$ (using ax_4).
2. From the facts $AC \cong AA$ it holds that $A = C$ (using ax_3).
3. There exist a point F where $bet(A, A, F)$ and $AF \cong BB$ (using ax_4).
4. From the facts $AF \cong BB$ it holds that $A = F$ (using ax_3).
5. From the facts $AF \cong BB$ and $A = C$ and $A = F$ it holds that $AA \cong BB$.
6. From the facts $AA \cong BB$ we proved a conjecture.

QED

Theorem 7 (th_2_11.) *Assuming that $bet(A, B, C)$ and $bet(D, E, F)$ and $AB \cong DE$ and $BC \cong EF$ it holds that $AC \cong DF$.*

Theorem 8 (th_2_12.) *Assuming that $D \neq A$ and $bet(D, A, E)$ and $AE \cong BC$ and $bet(D, A, F)$ and $AF \cong BC$ it holds that $E = F$.*

Chapter 3

Simple Properties of Betweenness

Theorem 9 (th_3_1.) *It holds that $bet(A, B, B)$.*

   Proof:
1. There exist a point G where $bet(A, B, G)$ and $BG \cong AA$ (using ax_4).
2. From the facts $BG \cong AA$ it holds that $B = G$ (using ax_3).
3. From the facts $bet(A, B, G)$ and $B = G$ it holds that $bet(A, B, B)$.
4. From the facts $bet(A, B, B)$ we proved a conjecture.
QED

Theorem 10 (th_3_2.) *Assuming that $bet(A, B, C)$ it holds that $bet(C, B, A)$.*

   Proof:
1. It holds that $bet(A, B, B)$ (using th_3_1).
2. It holds that $bet(B, B, B)$ (using th_3_1).
3. It holds that $bet(B, C, C)$ (using th_3_1).
4. From the facts $bet(A, B, B)$ and $bet(B, B, B)$ there exist a point F where $bet(B, F, B)$ and $bet(B, F, A)$ (using ax_7).
5. From the facts $bet(B, F, B)$ it holds that $B = F$ (using ax_6).
6. From the facts $bet(A, B, C)$ and $bet(B, C, C)$ there exist a point I where $bet(B, I, B)$ and $bet(C, I, A)$ (using ax_7).
7. From the facts $bet(B, I, B)$ it holds that $B = I$ (using ax_6).
8. From the facts $bet(C, I, A)$ and $B = F$ and $B = I$ it holds that $bet(C, B, A)$.
9. From the facts $bet(C, B, A)$ we proved a conjecture.
QED

Theorem 11 (th_3_3.) *It holds that $bet(A, A, B)$.*

   Proof:

1. It holds that $bet(B, A, A)$ (using th_3_1).
2. From the facts $bet(B, A, A)$ it holds that $bet(A, A, B)$ (using th_3_2).
3. From the facts $bet(A, A, B)$ we proved a conjecture.

QED
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Theorem 12 (th_3_4.) *Assuming that $bet(A, B, C)$ and $bet(B, A, C)$ it holds that $A = B$.*

   Proof:

1. It holds that $A = B$ or $A \neq B$ (using ax_g1).
   2. Assume that: $A = B$.
      3. From the facts $A = B$ we proved a conjecture.
   4. Assume that: $A \neq B$.
      5. From the facts $bet(A, B, C)$ and $bet(B, A, C)$ there exist a point E where $bet(B, E, B)$ and $bet(A, E, A)$ (using ax_7).
      6. From the facts $bet(A, E, A)$ it holds that $A = E$ (using ax_6).
      7. From the facts $bet(B, E, B)$ and $A = E$ it holds that $bet(B, A, B)$.
      8. From the facts $bet(B, A, B)$ it holds that $B = A$ (using ax_6).
      9. From the facts $A \neq B$ it holds that $B \neq A$.
      10. From the facts $B = A$ and $B \neq A$ we get contradiction.

QED
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Theorem 13 (th_3_5.) *Assuming that $bet(A, B, D)$ and $bet(B, C, D)$ it holds that $bet(A, B, C)$ and $bet(A, C, D)$.*

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Theorem 14 (th_3_6.) *Assuming that $bet(A, B, C)$ and $bet(A, C, D)$ it holds that $bet(B, C, D)$ and $bet(A, B, D)$.*

   Proof:

1. From the facts $bet(A, B, C)$ it holds that $bet(C, B, A)$ (using th_3_2).
2. From the facts $bet(A, C, D)$ it holds that $bet(D, C, A)$ (using th_3_2).
3. From the facts $bet(D, C, A)$ and $bet(C, B, A)$ it holds that $bet(D, C, B)$ and $bet(D, B, A)$ (using th_3_5).
4. From the facts $bet(D, B, A)$ it holds that $bet(A, B, D)$ (using th_3_2).
5. From the facts $bet(D, C, B)$ it holds that $bet(B, C, D)$ (using th_3_2).
6. From the facts $bet(B, C, D)$ and $bet(A, B, D)$ we proved a conjecture.

QED
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Theorem 15 (th_3_7.) *Assuming that $bet(A, B, C)$ and $bet(B, C, D)$ and $B \neq C$ it holds that $bet(A, C, D)$ and $bet(A, B, D)$.*

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Theorem 16 (th_3_9.) *Assuming that $bet4(A, B, C, D)$ it holds that $bet4(D, C, B, A)$.*

Proof:

1. From the facts $bet4(A, B, C, D)$ it holds that $bet(A, B, C)$ and $bet(A, B, D)$ and $bet(A, C, D)$ and $bet(B, C, D)$ (using $ax\_3\_8\_1$).

2. From the facts $bet(A, B, C)$ it holds that $bet(C, B, A)$ (using th_3_2).

3. From the facts $bet(A, B, D)$ it holds that $bet(D, B, A)$ (using th_3_2).

4. From the facts $bet(A, C, D)$ it holds that $bet(D, C, A)$ (using th_3_2).

5. From the facts $bet(B, C, D)$ it holds that $bet(D, C, B)$ (using th_3_2).

6. From the facts $bet(D, C, B)$ and $bet(D, C, A)$ and $bet(D, B, A)$ and $bet(C, B, A)$ it holds that $bet4(D, C, B, A)$ (using $ax\_3\_8\_2$).

7. From the facts $bet4(D, C, B, A)$ we proved a conjecture.

QED

Theorem 17 (th_3_10_1.) *Assuming that $bet4(A, B, C, D)$ it holds that $bet(A, B, C)$.*

Proof:

1. From the facts $bet4(A, B, C, D)$ it holds that $bet(A, B, C)$ and $bet(A, B, D)$ and $bet(A, C, D)$ and $bet(B, C, D)$ (using $ax\_3\_8\_1$).

2. From the facts $bet(A, B, C)$ we proved a conjecture.

QED

Theorem 18 (th_3_10_2.) *Assuming that $bet4(A, B, C, D)$ it holds that $bet(B, C, D)$.*

Proof:

1. From the facts $bet4(A, B, C, D)$ it holds that $bet(A, B, C)$ and $bet(A, B, D)$ and $bet(A, C, D)$ and $bet(B, C, D)$ (using $ax\_3\_8\_1$).

2. From the facts $bet(B, C, D)$ we proved a conjecture.

QED

Theorem 19 (th_3_11_1.) *Assuming that $bet(A, B, C)$ and $bet(A, D, B)$ it holds that $bet4(A, D, B, C)$.*

Proof:

1. From the facts $bet(A, D, B)$ and $bet(A, B, C)$ it holds that $bet(D, B, C)$ and $bet(A, D, C)$ (using th_3_6).

2. From the facts $bet(A, D, B)$ and $bet(A, D, C)$ and $bet(A, B, C)$ and $bet(D, B, C)$ it holds that $bet4(A, D, B, C)$ (using $ax\_3\_8\_2$).

3. From the facts $bet4(A, D, B, C)$ we proved a conjecture.

QED

Theorem 20 (th_3_11_2.) *Assuming that $bet(A, B, C)$ and $bet(B, D, C)$ it holds that $bet4(A, B, D, C)$.*

Proof:

1. From the facts $bet(A, B, C)$ and $bet(B, D, C)$ it holds that $bet(A, B, D)$ and $bet(A, D, C)$ (using th_3_5).

2. From the facts $bet(A, B, D)$ and $bet(A, B, C)$ and $bet(A, D, C)$ and $bet(B, D, C)$ it holds that $bet4(A, B, D, C)$ (using $ax\_3\_8\_2$).

3. From the facts $bet4(A, B, D, C)$ we proved a conjecture.

QED

Theorem 21 (th_3_12_1.) *Assuming that $bet(A, B, C)$ and $bet(B, C, D)$ and $B \neq C$ it holds that $bet4(A, B, C, D)$.*

   Proof:

1. From the facts $bet(A, B, C)$ and $bet(B, C, D)$ and $B \neq C$ it holds that $bet(A, C, D)$ and $bet(A, B, D)$ (using th_3_7).

2. From the facts $bet(A, C, D)$ and $bet(A, B, C)$ it holds that $bet4(A, B, C, D)$ (using $th\_3\_11\_1$).

3. From the facts $bet4(A, B, C, D)$ we proved a conjecture.

QED

Theorem 22 (th_3_12_2.) *Assuming that $bet(A, B, C)$ and $bet(A, C, D)$ it holds that $bet4(A, B, C, D)$.*

   Proof:

1. From the facts $bet(A, C, D)$ and $bet(A, B, C)$ it holds that $bet4(A, B, C, D)$ (using $th\_3\_11\_1$).

2. From the facts $bet4(A, B, C, D)$ we proved a conjecture.

QED

Theorem 23 (th_3_13.) *There exist point A, point B, such that $A \neq B$.*

   Proof:

1. There exist a point A and a point B and a point C where $not\_bet(A, B, C)$ and $not\_bet(B, C, A)$ and $not\_bet(C, A, B)$ (using ax_8).

2. There exist a point D where $bet(A, A, D)$ and $AD \cong AA$ (using ax_4).

3. It holds that $A = B$ or $A \neq B$ (using ax_g1).

   4. Assume that: $A = B$.

     5. It holds that $A = C$ or $A \neq C$ (using ax_g1).

       6. Assume that: $A = C$.

         7. It holds that $A = D$ or $A \neq D$ (using ax_g1).

           8. Assume that: $A = D$.

             9. From the facts $bet(A, A, D)$ and $A = B$ and $A = C$ and $A = D$ it holds that $bet(A, A, A)$.

             10. From the facts $bet(A, A, A)$ and $A = B$ and $A = C$ it holds that $bet(A, B, C)$.

             11. From the facts $not\_bet(A, B, C)$ and $bet(A, B, C)$ we get contradiction (using ax_false_bet).

           12. Assume that: $A \neq D$.

13. From the facts $A \neq D$ we proved a conjecture.

14. Assume that: $A \neq C$.

15. From the facts $A \neq C$ we proved a conjecture.

16. Assume that: $A \neq B$.

17. From the facts $A \neq B$ we proved a conjecture.

QED

Theorem 24 (th_3_14.) *There exist point C, such that bet(A, B, C) and $B \neq C$.*

Theorem 25 (th_3_15_1.) *Assuming that $A \neq B$ there exist point C, such that bet(A, B, C) and $A \neq C$ and $B \neq C$.*

Proof:

1. There exist a point D where $bet(A, B, D)$ and $B \neq D$ (using th_3_14).

2. It holds that $A = D$ or $A \neq D$ (using ax_g1).

   3. Assume that: $A = D$.

      4. From the facts $bet(A, B, D)$ and $A = D$ it holds that $bet(A, B, A)$.

      5. From the facts $bet(A, B, A)$ it holds that $A = B$ (using ax_6).

      6. From the facts $A = B$ and $A \neq B$ we get contradiction.

   7. Assume that: $A \neq D$.

      8. From the facts $bet(A, B, D)$ and $A \neq D$ and $B \neq D$ we proved a conjecture.

QED

Theorem 26 (th_3_15_2.) *Assuming that $A \neq B$ there exist point C, point D, such that bet4(A, B, C, D) and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$.*

Theorem 27 (th_3_17.) *Assuming that $bet(A, B, C)$ and $bet(D, E, C)$ and $bet(A, F, D)$ there exist point G, such that $bet(F, G, C)$ and $bet(B, G, E)$.*

Chapter 4

Simple Properties of Congruence and Betweenness

Theorem 28 (th_4_2.) *Assuming that $ifs(A, B, C, D, E, F, G, I)$ it holds that $BD \cong FI$.*

Theorem 29 (th_4_3.) *Assuming that $bet(A, B, C)$ and $bet(D, E, F)$ and $AC \cong DF$ and $BC \cong EF$ it holds that $AB \cong DE$.*

Theorem 30 (th_4_5.) *Assuming that $bet(A, B, C)$ and $AC \cong DE$ there exist point F, such that $bet(D, F, E)$ and $cong3(A, B, C, D, F, E)$.*

Theorem 31 (th_4_6.) *Assuming that $bet(A, B, C)$ and $cong3(A, B, C, D, E, F)$ it holds that $bet(D, E, F)$.*

Theorem 32 (th_4_11.) *Assuming that $col(A, B, C)$ it holds that $col(B, C, A)$ and $col(C, A, B)$ and $col(C, B, A)$ and $col(B, A, C)$ and $col(A, C, B)$.*

    Proof:
1. From the facts $col(A, B, C)$ it holds that $bet(A, B, C)$ or $bet(B, C, A)$ or $bet(C, A, B)$ (using $ax\_4\_10\_1$).
  2. Assume that: $bet(A, B, C)$.
    3. From the facts $bet(A, B, C)$ it holds that $col(C, A, B)$ (using $ax\_4\_10\_3$).
    4. From the facts $bet(A, B, C)$ it holds that $bet(C, B, A)$ (using th_3_2).
    5. From the facts $bet(C, B, A)$ it holds that $col(A, C, B)$ (using $ax\_4\_10\_3$).
    6. From the facts $bet(C, B, A)$ it holds that $col(C, B, A)$ (using $ax\_4\_10\_2$).
    7. From the facts $bet(A, B, C)$ it holds that $col(B, C, A)$ (using $ax\_4\_10\_4$).

8. From the facts $bet(C, B, A)$ it holds that $col(B, A, C)$ (using $ax\_4\_10\_4$).

9. From the facts $col(B, C, A)$ and $col(C, A, B)$ and $col(C, B, A)$ and $col(B, A, C)$ and $col(A, C, B)$ we proved a conjecture.

10. Assume that: $bet(B, C, A)$.

11. From the facts $bet(B, C, A)$ it holds that $bet(A, C, B)$ (using th_3_2).

12. From the facts $bet(A, C, B)$ it holds that $col(B, A, C)$ (using $ax\_4\_10\_3$).

13. From the facts $bet(A, C, B)$ it holds that $col(A, C, B)$ (using $ax\_4\_10\_2$).

14. From the facts $bet(B, C, A)$ it holds that $col(B, C, A)$ (using $ax\_4\_10\_2$).

15. From the facts $bet(A, C, B)$ it holds that $col(C, B, A)$ (using $ax\_4\_10\_4$).

16. From the facts $bet(B, C, A)$ it holds that $col(C, A, B)$ (using $ax\_4\_10\_4$).

17. From the facts $col(B, C, A)$ and $col(C, A, B)$ and $col(C, B, A)$ and $col(B, A, C)$ and $col(A, C, B)$ we proved a conjecture.

18. Assume that: $bet(C, A, B)$.

19. From the facts $bet(C, A, B)$ it holds that $col(B, C, A)$ (using $ax\_4\_10\_3$).

20. From the facts $bet(C, A, B)$ it holds that $bet(B, A, C)$ (using th_3_2).

21. From the facts $bet(B, A, C)$ it holds that $col(C, B, A)$ (using $ax\_4\_10\_3$).

22. From the facts $bet(B, A, C)$ it holds that $col(B, A, C)$ (using $ax\_4\_10\_2$).

23. From the facts $bet(C, A, B)$ it holds that $col(C, A, B)$ (using $ax\_4\_10\_2$).

24. From the facts $bet(B, A, C)$ it holds that $col(A, C, B)$ (using $ax\_4\_10\_4$).

25. From the facts $col(B, C, A)$ and $col(C, A, B)$ and $col(C, B, A)$ and $col(B, A, C)$ and $col(A, C, B)$ we proved a conjecture.

QED

Theorem 33 (th_4_12.) *It holds that $col(A, A, B)$.*

Proof:

1. It holds that $bet(A, A, B)$ (using th_3_3).

2. From the facts $bet(A, A, B)$ it holds that $col(A, A, B)$ (using $ax\_4\_10\_2$).

3. From the facts $col(A, A, B)$ we proved a conjecture.

QED

Theorem 34 (th_4_13.) *Assuming that $col(A, B, C)$ and $cong3(A, B, C, D, E, F)$ it holds that $col(D, E, F)$.*

Proof:

1. From the facts $cong3(A, B, C, D, E, F)$ it holds that $AB \cong DE$ and $AC \cong DF$ and $BC \cong EF$ (using $ax\_4\_4\_1$).

2. From the facts $AB \cong DE$ it holds that $AB \cong ED$ (using th_2_5).

3. From the facts $AB \cong ED$ it holds that $BA \cong ED$ (using th_2_4).

4. From the facts $BC \cong EF$ it holds that $BC \cong FE$ (using th_2_5).

5. From the facts $BC \cong FE$ it holds that $CB \cong FE$ (using th_2_4).

6. From the facts $AC \cong DF$ and $AB \cong DE$ and $CB \cong FE$ it holds that $cong3(A, C, B, D, F, E)$ (using $ax\_4\_4\_2$).

7. From the facts $BA \cong ED$ and $BC \cong EF$ and $AC \cong DF$ it holds that $cong3(B, A, C, E, D, F)$ (using $ax\_4\_4\_2$).

8. From the facts $col(A, B, C)$ it holds that $bet(A, B, C)$ or $bet(B, C, A)$ or $bet(C, A, B)$ (using $ax\_4\_10\_1$).

   9. Assume that: $bet(A, B, C)$.

   10. From the facts $bet(A, B, C)$ and $cong3(A, B, C, D, E, F)$ it holds that $bet(D, E, F)$ (using th_4_6).

   11. From the facts $bet(D, E, F)$ it holds that $col(D, E, F)$ (using $ax\_4\_10\_2$).

   12. From the facts $col(D, E, F)$ we proved a conjecture.

  13. Assume that: $bet(B, C, A)$.

   14. From the facts $bet(B, C, A)$ it holds that $bet(A, C, B)$ (using th_3_2).

   15. From the facts $bet(A, C, B)$ and $cong3(A, C, B, D, F, E)$ it holds that $bet(D, F, E)$ (using th_4_6).

   16. From the facts $bet(D, F, E)$ it holds that $col(D, F, E)$ (using $ax\_4\_10\_2$).

   17. From the facts $col(D, F, E)$ it holds that $col(F, E, D)$ and $col(E, D, F)$ and $col(E, F, D)$ and $col(F, D, E)$ and $col(D, E, F)$ (using th_4_11).

   18. From the facts $col(D, E, F)$ we proved a conjecture.

  19. Assume that: $bet(C, A, B)$.

   20. From the facts $bet(C, A, B)$ it holds that $bet(B, A, C)$ (using th_3_2).

   21. From the facts $bet(B, A, C)$ and $cong3(B, A, C, E, D, F)$ it holds that $bet(E, D, F)$ (using th_4_6).

   22. From the facts $bet(E, D, F)$ it holds that $col(E, D, F)$ (using $ax\_4\_10\_2$).

   23. From the facts $col(E, D, F)$ it holds that $col(D, F, E)$ and $col(F, E, D)$ and $col(F, D, E)$ and $col(D, E, F)$ and $col(E, F, D)$ (using th_4_11).

   24. From the facts $col(D, E, F)$ we proved a conjecture.

QED

Theorem 35 (th_4_14.) *Assuming that $col(A, B, C)$ and $AB \cong DE$ there exist point F, such that $cong3(A, B, C, D, E, F)$.*

Theorem 36 (th_4_16.) *Assuming that $fs(A, B, C, D, E, F, G, I)$ and $A \neq B$ it holds that $CD \cong GI$.*

Theorem 37 (th_4_17.) *Assuming that $A \neq B$ and $col(A, B, C)$ and $AD \cong AE$ and $BD \cong BE$ it holds that $CD \cong CE$.*

Theorem 38 (th_4_18.) *Assuming that $A \neq B$ and $col(A, B, C)$ and $AC \cong AD$ and $BC \cong BD$ it holds that $C = D$.*

   Proof:

1. From the facts $A \neq B$ and $col(A, B, C)$ and $AC \cong AD$ and $BC \cong BD$ it holds that $CC \cong CD$ (using th_4_17).

2. From the facts $CC \cong CD$ it holds that $CD \cong CC$ (using th_2_2).
3. From the facts $CD \cong CC$ it holds that $C = D$ (using ax_3).
4. From the facts $C = D$ we proved a conjecture.

QED

Theorem 39 (th_4_19.) *Assuming that $bet(A, B, C)$ and $AB \cong AD$ and $CB \cong CD$ it holds that $B = D$.*

    Proof:

1. It holds that $bet(B, A, A)$ (using th_3_1).
2. From the facts $bet(A, B, C)$ it holds that $col(C, A, B)$ (using $ax\_4\_10\_3$).
3. From the facts $AB \cong AD$ it holds that $AD \cong AB$ (using th_2_2).
4. It holds that $A = B$ or $A \neq B$ (using ax_g1).
  5. Assume that: $A = B$.
    6. From the facts $AD \cong AB$ and $A = B$ it holds that $AD \cong AA$.
    7. From the facts $AD \cong AA$ it holds that $A = D$ (using ax_3).
    8. From the facts $A = B$ and $A = D$ it holds that $B = D$.
    9. From the facts $B = D$ we proved a conjecture.
 10. Assume that: $A \neq B$.
   11. It holds that $A = C$ or $A \neq C$ (using ax_g1).
    12. Assume that: $A = C$.
     13. From the facts $bet(A, B, C)$ and $A = C$ it holds that $bet(A, B, A)$.
     14. From the facts $bet(A, B, A)$ and $bet(B, A, A)$ it holds that $A = B$ (using th_3_4).
     15. From the facts $A = B$ and $A \neq B$ we get contradiction.
    16. Assume that: $A \neq C$.
     17. From the facts $A \neq C$ it holds that $C \neq A$.
     18. From the facts $C \neq A$ and $col(C, A, B)$ and $CB \cong CD$ and $AB \cong AD$ it holds that $B = D$ (using th_4_18).
     19. From the facts $B = D$ we proved a conjecture.

QED

Chapter 5

Relationship between Congruence and Comparison of Distances

Theorem 40 (th_5_1.) *Assuming that $A \neq B$ and $bet(A, B, C)$ and $bet(A, B, D)$ it holds that $bet(A, C, D)$ or $bet(A, D, C)$.*

Theorem 41 (th_5_2.) *Assuming that $A \neq B$ and $bet(A, B, C)$ and $bet(A, B, D)$ it holds that $bet(B, C, D)$ or $bet(B, D, C)$.*

Proof:

1. From the facts $A \neq B$ and $bet(A, B, C)$ and $bet(A, B, C)$ it holds that $bet(A, C, C)$ or $bet(A, C, C)$ (using th_5_1).

  2. Assume that: $bet(A, C, C)$.

    3. From the facts $bet(A, B, C)$ and $bet(A, C, C)$ it holds that $bet(B, C, C)$ and $bet(A, B, C)$ (using th_3_6).

    4. From the facts $bet(A, C, C)$ and $bet(A, C, C)$ it holds that $bet(C, C, C)$ and $bet(A, C, C)$ (using th_3_6).

    5. From the facts $A \neq B$ and $bet(A, B, C)$ and $bet(A, B, D)$ it holds that $bet(A, C, D)$ or $bet(A, D, C)$ (using th_5_1).

      6. Assume that: $bet(A, C, D)$.

        7. From the facts $bet(A, B, C)$ and $bet(A, C, D)$ it holds that $bet(B, C, D)$ and $bet(A, B, D)$ (using th_3_6).

        8. From the facts $bet(B, C, D)$ we proved a conjecture.

      9. Assume that: $bet(A, D, C)$.

      10. From the facts $bet(A, B, D)$ and $bet(A, D, C)$ it holds that $bet(B, D, C)$ and $bet(A, B, C)$ (using th_3_6).

      11. From the facts $bet(B, D, C)$ we proved a conjecture.

12. Assume that: $bet(A, C, C)$.

13. From the facts $bet(A, B, C)$ and $bet(A, C, C)$ it holds that $bet(B, C, C)$ and $bet(A, B, C)$ (using th_3_6).

14. From the facts $A \neq B$ and $bet(A, B, C)$ and $bet(A, B, D)$ it holds that $bet(A, C, D)$ or $bet(A, D, C)$ (using th_5_1).

15. Assume that: $bet(A, C, D)$.

16. From the facts $bet(A, B, C)$ and $bet(A, C, D)$ it holds that $bet(B, C, D)$ and $bet(A, B, D)$ (using th_3_6).

17. From the facts $bet(B, C, D)$ we proved a conjecture.

18. Assume that: $bet(A, D, C)$.

19. From the facts $bet(A, B, D)$ and $bet(A, D, C)$ it holds that $bet(B, D, C)$ and $bet(A, B, C)$ (using th_3_6).

20. From the facts $bet(B, D, C)$ we proved a conjecture.

QED

Theorem 42 (th_5_3.) *Assuming that $bet(A, B, D)$ and $bet(A, C, D)$ it holds that $bet(A, B, C)$ or $bet(A, C, B)$.*

Theorem 43 (th_5_5_1.) *Assuming that $le(A, B, C, D)$ there exist point E, such that $bet(A, B, E)$ and $AE \cong CD$.*

Theorem 44 (th_5_5_2.) *Assuming that $bet(A, B, E)$ and $AE \cong CD$ it holds that $le(A, B, C, D)$.*

Theorem 45 (th_5_6.) *Assuming that $le(A, B, C, D)$ and $AB \cong EF$ and $CD \cong GI$ it holds that $le(E, F, G, I)$.*

Theorem 46 (th_5_7.) *It holds that $le(A, B, A, B)$.*

   Proof:

1. It holds that $bet(A, B, B)$ (using th_3_1).

2. It holds that $AB \cong AB$ (using th_2_1).

3. From the facts $bet(A, B, B)$ and $AB \cong AB$ it holds that $le(A, B, A, B)$ (using $ax\_5\_4\_2$).

4. From the facts $le(A, B, A, B)$ we proved a conjecture.

QED

Theorem 47 (th_5_8.) *Assuming that $le(A, B, C, D)$ and $le(C, D, E, F)$ it holds that $le(A, B, E, F)$.*

Theorem 48 (th_5_9.) *Assuming that $le(A, B, C, D)$ and $le(C, D, A, B)$ it holds that $AB \cong CD$.*

Theorem 49 (th_5_10.) *It holds that* $le(A, B, C, D)$ *or* $le(C, D, A, B)$.

Theorem 50 (th_5_11.) *It holds that* $le(A, A, B, C)$.

Proof:
1. It holds that $bet(B, B, C)$ (using th_3_3).
2. It holds that $AA \cong BB$ (using th_2_8).
3. From the facts $bet(B, B, C)$ and $AA \cong BB$ it holds that $le(A, A, B, C)$ (using $ax\_5\_4\_2$).
4. From the facts $le(A, A, B, C)$ we proved a conjecture.
QED

Theorem 51 (th_5_12_1.) *Assuming that* $col(A, B, C)$ *and* $bet(A, B, C)$ *it holds that* $le(A, B, A, C)$ *and* $le(B, C, A, C)$.

Proof:
1. From the facts $bet(A, B, C)$ it holds that $bet(C, B, A)$ (using th_3_2).
2. It holds that $bet(A, C, C)$ (using th_3_1).
3. It holds that $bet(B, A, A)$ (using th_3_1).
4. It holds that $AB \cong BA$ (using ax_1).
5. It holds that $BA \cong AB$ (using ax_1).
6. It holds that $BC \cong CB$ (using ax_1).
7. It holds that $CA \cong AC$ (using ax_1).
8. From the facts $bet(A, B, C)$ and $BA \cong AB$ it holds that $le(B, A, A, C)$ (using $ax\_5\_4\_2$).
9. From the facts $bet(A, C, C)$ and $CA \cong AC$ it holds that $le(C, A, A, C)$ (using $ax\_5\_4\_2$).
10. From the facts $bet(B, A, A)$ and $AB \cong BA$ it holds that $le(A, B, B, A)$ (using $ax\_5\_4\_2$).
11. From the facts $bet(C, B, A)$ and $BC \cong CB$ it holds that $le(B, C, C, A)$ (using $ax\_5\_4\_2$).
12. From the facts $le(A, B, B, A)$ and $le(B, A, A, C)$ it holds that $le(A, B, A, C)$ (using th_5_8).
13. From the facts $le(B, C, C, A)$ and $le(C, A, A, C)$ it holds that $le(B, C, A, C)$ (using th_5_8).
14. From the facts $le(A, B, A, C)$ and $le(B, C, A, C)$ we proved a conjecture.
QED

Theorem 52 (th_5_12_2.) *Assuming that* $col(A, B, C)$ *and* $le(A, B, A, C)$ *and* $le(B, C, A, C)$ *it holds that* $bet(A, B, C)$.

Proof:
1. It holds that $bet(A, A, B)$ (using th_3_3).

2. It holds that $bet(A, A, C)$ (using th_3_3).

3. It holds that $bet(B, B, A)$ (using th_3_3).

4. It holds that $bet(A, B, B)$ (using th_3_1).

5. It holds that $bet(A, C, C)$ (using th_3_1).

6. It holds that $bet(B, A, A)$ (using th_3_1).

7. From the facts $bet(A, A, C)$ it holds that $col(A, A, C)$ (using $ax\_4\_10\_2$).

8. From the facts $bet(A, C, C)$ it holds that $col(A, C, C)$ (using $ax\_4\_10\_2$).

9. From the facts $bet(B, A, A)$ it holds that $col(B, A, A)$ (using $ax\_4\_10\_2$).

10. From the facts $bet(B, B, A)$ it holds that $col(B, B, A)$ (using $ax\_4\_10\_2$).

11. From the facts $col(A, A, C)$ and $bet(A, A, C)$ it holds that $le(A, A, A, C)$ and $le(A, C, A, C)$ (using $th\_5\_12\_1$).

12. From the facts $col(A, C, C)$ and $bet(A, C, C)$ it holds that $le(A, C, A, C)$ and $le(C, C, A, C)$ (using $th\_5\_12\_1$).

13. From the facts $col(B, A, A)$ and $bet(B, A, A)$ it holds that $le(B, A, B, A)$ and $le(A, A, B, A)$ (using $th\_5\_12\_1$).

14. From the facts $col(B, B, A)$ and $bet(B, B, A)$ it holds that $le(B, B, B, A)$ and $le(B, A, B, A)$ (using $th\_5\_12\_1$).

15. From the facts $le(A, C, A, C)$ and $le(A, C, A, C)$ it holds that $AC \cong AC$ (using th_5_9).

16. From the facts $le(B, A, B, A)$ and $le(B, A, B, A)$ it holds that $BA \cong BA$ (using th_5_9).

17. It holds that $BA \cong AB$ (using ax_1).

18. From the facts $col(A, B, C)$ it holds that $bet(A, B, C)$ or $bet(B, C, A)$ or $bet(C, A, B)$ (using $ax\_4\_10\_1$).

  19. Assume that: $bet(A, B, C)$.

    20. From the facts $bet(A, B, C)$ we proved a conjecture.

  21. Assume that: $bet(B, C, A)$.

    22. From the facts $bet(B, C, A)$ it holds that $bet(A, C, B)$ (using th_3_2).

    23. From the facts $bet(A, C, B)$ it holds that $col(A, C, B)$ (using $ax\_4\_10\_2$).

    24. From the facts $col(A, C, B)$ and $bet(A, C, B)$ it holds that $le(A, C, A, B)$ and $le(C, B, A, B)$ (using $th\_5\_12\_1$).

    25. From the facts $le(A, C, A, B)$ and $le(A, B, A, C)$ it holds that $AC \cong AB$ (using th_5_9).

    26. From the facts $AC \cong AB$ it holds that $CA \cong AB$ (using th_2_4).

    27. From the facts $bet(B, C, A)$ and $bet(A, A, B)$ and $BA \cong AB$ and $CA \cong AB$ it holds that $BC \cong AA$ (using th_4_3).

    28. From the facts $BC \cong AA$ it holds that $B = C$ (using ax_3).

    29. From the facts $bet(A, B, B)$ and $B = C$ it holds that $bet(A, B, C)$.

    30. From the facts $bet(A, B, C)$ we proved a conjecture.

  31. Assume that: $bet(C, A, B)$.

    32. From the facts $bet(C, A, B)$ it holds that $bet(B, A, C)$ (using th_3_2).

33. From the facts $bet(B, A, C)$ it holds that $col(B, A, C)$ (using $ax\_4\_10\_2$).

34. From the facts $col(B, A, C)$ and $bet(B, A, C)$ it holds that $le(B, A, B, C)$ and $le(A, C, B, C)$ (using $th\_5\_12\_1$).

35. From the facts $le(A, C, B, C)$ and $le(B, C, A, C)$ it holds that $AC \cong BC$ (using th_5_9).

36. From the facts $bet(A, A, C)$ and $bet(B, A, C)$ and $AC \cong BC$ and $AC \cong AC$ it holds that $AA \cong BA$ (using th_4_3).

37. From the facts $bet(B, A, A)$ and $bet(B, B, A)$ and $BA \cong BA$ and $AA \cong BA$ it holds that $BA \cong BB$ (using th_4_3).

38. From the facts $BA \cong BB$ it holds that $B = A$ (using ax_3).

39. From the facts $bet(A, A, C)$ and $B = A$ it holds that $bet(B, B, C)$.

40. From the facts $bet(B, B, C)$ and $B = A$ it holds that $bet(A, B, C)$.

41. From the facts $bet(A, B, C)$ we proved a conjecture.

QED

Chapter 6

Half-lines and Lines

Theorem 53 (th_6_2_1.) *Assuming that $A \neq B$ and $C \neq B$ and $D \neq B$ and $bet(A, B, D)$ and $bet(C, B, D)$ it holds that $out(B, A, C)$.*

Theorem 54 (th_6_2_2.) *Assuming that $A \neq D$ and $B \neq D$ and $C \neq D$ and $bet(A, D, C)$ and $out(D, A, B)$ it holds that $bet(B, D, C)$.*

Theorem 55 (th_6_3_1.) *Assuming that $out(C, A, B)$ there exist point D, such that $A \neq C$ and $B \neq C$ and $D \neq C$ and $bet(A, C, D)$ and $bet(B, C, D)$.*

Theorem 56 (th_6_3_2.) *Assuming that $A \neq B$ and $C \neq B$ and $D \neq B$ and $bet(A, B, D)$ and $bet(C, B, D)$ it holds that $out(B, A, C)$.*

    Proof:
1. From the facts $A \neq B$ and $C \neq B$ and $D \neq B$ and $bet(A, B, D)$ and $bet(C, B, D)$ it holds that $out(B, A, C)$ (using $th\_6\_2\_1$).
2. From the facts $out(B, A, C)$ we proved a conjecture.
QED

Theorem 57 (th_6_4_1.) *Assuming that $out(D, A, B)$ it holds that $col(A, D, B)$ and $not\_bet(A, D, B)$.*

Theorem 58 (th_6_4_2.) *Assuming that $col(A, B, C)$ and $not\_bet(A, B, C)$ it holds that $out(B, A, C)$.*

    Proof:
1. It holds that $bet(A, B, B)$ (using th_3_1).
2. It holds that $bet(A, A, C)$ (using th_3_3).
3. From the facts $col(A, B, C)$ it holds that $bet(A, B, C)$ or $bet(B, C, A)$ or $bet(C, A, B)$ (using $ax\_4\_10\_1$).

4. Assume that: $bet(A, B, C)$.

  5. From the facts $bet(A, B, C)$ and $not\_bet(A, B, C)$ we get contradiction (using ax_false_bet).

6. Assume that: $bet(B, C, A)$.

  7. It holds that $A = B$ or $A \neq B$ (using ax_g1).

    8. Assume that: $A = B$.

      9. From the facts $bet(A, A, C)$ and $A = B$ it holds that $bet(A, B, C)$.

      10. From the facts $not\_bet(A, B, C)$ and $bet(A, B, C)$ we get contradiction (using ax_false_bet).

    11. Assume that: $A \neq B$.

      12. It holds that $B = C$ or $B \neq C$ (using ax_g1).

        13. Assume that: $B = C$.

          14. From the facts $bet(A, B, B)$ and $B = C$ it holds that $bet(A, B, C)$.

          15. From the facts $not\_bet(A, B, C)$ and $bet(A, B, C)$ we get contradiction (using ax_false_bet).

        16. Assume that: $B \neq C$.

          17. From the facts $B \neq C$ it holds that $C \neq B$.

          18. From the facts $A \neq B$ and $C \neq B$ and $bet(B, C, A)$ it holds that $out(B, A, C)$ (using $ax\_6\_1\_3$).

          19. From the facts $out(B, A, C)$ we proved a conjecture.

20. Assume that: $bet(C, A, B)$.

  21. From the facts $bet(C, A, B)$ it holds that $bet(B, A, C)$ (using th_3_2).

  22. It holds that $A = B$ or $A \neq B$ (using ax_g1).

    23. Assume that: $A = B$.

      24. From the facts $bet(A, A, C)$ and $A = B$ it holds that $bet(A, B, C)$.

      25. From the facts $not\_bet(A, B, C)$ and $bet(A, B, C)$ we get contradiction (using ax_false_bet).

    26. Assume that: $A \neq B$.

      27. It holds that $B = C$ or $B \neq C$ (using ax_g1).

        28. Assume that: $B = C$.

          29. From the facts $bet(A, B, B)$ and $B = C$ it holds that $bet(A, B, C)$.

          30. From the facts $not\_bet(A, B, C)$ and $bet(A, B, C)$ we get contradiction (using ax_false_bet).

        31. Assume that: $B \neq C$.

          32. From the facts $B \neq C$ it holds that $C \neq B$.

          33. From the facts $A \neq B$ and $C \neq B$ and $bet(B, A, C)$ it holds that $out(B, A, C)$ (using $ax\_6\_1\_2$).

          34. From the facts $out(B, A, C)$ we proved a conjecture.

QED

Theorem 59 (th_6_5.) *Assuming that $A \neq B$ it holds that $out(B, A, A)$.*

Proof:

1. It holds that $bet(B, A, A)$ (using th_3_1).
2. From the facts $A \neq B$ and $A \neq B$ and $bet(B, A, A)$ it holds that $out(B, A, A)$ (using $ax\_6\_1\_3$).
3. From the facts $out(B, A, A)$ we proved a conjecture.

QED

Theorem 60 (th_6_6.) *Assuming that $out(A, B, C)$ it holds that $out(A, C, B)$.*

Proof:

1. It holds that $A = B$ or $A \neq B$ (using ax_g1).
  2. Assume that: $A = B$.
    3. From the facts $out(A, B, C)$ and $A = B$ it holds that $out(A, A, C)$.
    4. From the facts $out(A, A, C)$ there exist a point I where $A \neq A$ and $C \neq A$ and $I \neq A$ and $bet(A, A, I)$ and $bet(C, A, I)$ (using $th\_6\_3\_1$).
    5. From the facts $A \neq A$ we get contradiction.
  6. Assume that: $A \neq B$.
    7. It holds that $A = C$ or $A \neq C$ (using ax_g1).
      8. Assume that: $A = C$.
        9. From the facts $out(A, B, C)$ and $A = C$ it holds that $out(A, B, A)$.
        10. From the facts $out(A, B, A)$ there exist a point I where $B \neq A$ and $A \neq A$ and $I \neq A$ and $bet(B, A, I)$ and $bet(A, A, I)$ (using $th\_6\_3\_1$).
        11. From the facts $A \neq A$ we get contradiction.
      12. Assume that: $A \neq C$.
        13. There exist a point G where $bet(B, A, G)$ and $A \neq G$ (using th_3_14).
        14. From the facts $A \neq B$ it holds that $B \neq A$.
        15. From the facts $A \neq C$ it holds that $C \neq A$.
        16. From the facts $A \neq G$ it holds that $G \neq A$.
        17. From the facts $B \neq A$ and $C \neq A$ and $G \neq A$ and $bet(B, A, G)$ and $out(A, B, C)$ it holds that $bet(C, A, G)$ (using $th\_6\_2\_2$).
        18. From the facts $A \neq C$ it holds that $C \neq A$.
        19. From the facts $A \neq B$ it holds that $B \neq A$.
        20. From the facts $A \neq G$ it holds that $G \neq A$.
        21. From the facts $C \neq A$ and $B \neq A$ and $G \neq A$ and $bet(C, A, G)$ and $bet(B, A, G)$ it holds that $out(A, C, B)$ (using $th\_6\_3\_2$).
        22. From the facts $out(A, C, B)$ we proved a conjecture.

QED

Theorem 61 (th_6_7.) *Assuming that $out(A, B, C)$ and $out(A, C, D)$ it holds that $out(A, B, D)$.*

Proof:

1. It holds that $A = B$ or $A \neq B$ (using ax_g1).

2. Assume that: $A = B$.

  3. From the facts $out(A, B, C)$ and $A = B$ it holds that $out(A, A, C)$.

  4. From the facts $out(A, A, C)$ there exist a point N where $A \neq A$ and $C \neq A$ and $N \neq A$ and $bet(A, A, N)$ and $bet(C, A, N)$ (using *th_6_3_1*).

  5. From the facts $A \neq A$ we get contradiction.

6. Assume that: $A \neq B$.

  7. It holds that $A = C$ or $A \neq C$ (using *ax_g1*).

    8. Assume that: $A = C$.

      9. From the facts $out(A, C, D)$ and $A = C$ it holds that $out(A, A, D)$.

      10. From the facts $out(A, A, D)$ there exist a point P where $A \neq A$ and $D \neq A$ and $P \neq A$ and $bet(A, A, P)$ and $bet(D, A, P)$ (using *th_6_3_1*).

      11. From the facts $A \neq A$ we get contradiction.

  12. Assume that: $A \neq C$.

    13. It holds that $A = D$ or $A \neq D$ (using *ax_g1*).

      14. Assume that: $A = D$.

        15. There exist a point I where $bet(B, A, I)$ and $A \neq I$ (using *th_3_14*).

        16. From the facts $A \neq B$ it holds that $B \neq A$.

        17. From the facts $out(A, C, D)$ and $A = D$ it holds that $out(A, C, A)$.

        18. From the facts $out(A, C, A)$ there exist a point N where $C \neq A$ and $A \neq A$ and $N \neq A$ and $bet(C, A, N)$ and $bet(A, A, N)$ (using *th_6_3_1*).

        19. From the facts $A \neq A$ we get contradiction.

      20. Assume that: $A \neq D$.

        21. There exist a point J where $bet(B, A, J)$ and $A \neq J$ (using *th_3_14*).

        22. From the facts $A \neq B$ it holds that $B \neq A$.

        23. From the facts $A \neq C$ it holds that $C \neq A$.

        24. From the facts $A \neq J$ it holds that $J \neq A$.

        25. From the facts $B \neq A$ and $C \neq A$ and $J \neq A$ and $bet(B, A, J)$ and $out(A, B, C)$ it holds that $bet(C, A, J)$ (using *th_6_2_2*).

        26. From the facts $A \neq C$ it holds that $C \neq A$.

        27. From the facts $A \neq D$ it holds that $D \neq A$.

        28. From the facts $A \neq J$ it holds that $J \neq A$.

        29. From the facts $C \neq A$ and $D \neq A$ and $J \neq A$ and $bet(C, A, J)$ and $out(A, C, D)$ it holds that $bet(D, A, J)$ (using *th_6_2_2*).

        30. From the facts $A \neq B$ it holds that $B \neq A$.

        31. From the facts $A \neq D$ it holds that $D \neq A$.

        32. From the facts $A \neq J$ it holds that $J \neq A$.

        33. From the facts $B \neq A$ and $D \neq A$ and $J \neq A$ and $bet(B, A, J)$ and $bet(D, A, J)$ it holds that $out(A, B, D)$ (using *th_6_3_2*).

34. From the facts $out(A, B, D)$ we proved a conjecture.

QED

Theorem 62 (th_6_11_1.) *Assuming that $A \neq B$ and $C \neq D$ there exist point E, such that $out(B, E, A)$ and $BE \cong CD$.*

Theorem 63 (th_6_11_2.) *Assuming that $A \neq B$ and $C \neq D$ and $out(B, E, A)$ and $BE \cong CD$ and $out(B, F, A)$ and $BF \cong CD$ it holds that $E = F$.*

Theorem 64 (th_6_13_1.) *Assuming that $out(C, A, B)$ and $le(C, A, C, B)$ it holds that $bet(C, A, B)$.*

Theorem 65 (th_6_13_2.) *Assuming that $out(A, B, C)$ and $bet(A, B, C)$ it holds that $le(A, B, A, C)$.*

   Proof:

1. From the facts $bet(A, B, C)$ it holds that $col(A, B, C)$ (using $ax\_4\_10\_2$).

2. From the facts $col(A, B, C)$ and $bet(A, B, C)$ it holds that $le(A, B, A, C)$ and $le(B, C, A, C)$ (using $th\_5\_12\_1$).

3. From the facts $le(A, B, A, C)$ we proved a conjecture.

QED

Theorem 66 (th_6_15_1.) *Assuming that $A \neq B$ and $A \neq C$ and $bet(B, A, C)$ and $point\_on\_line(D, A, B)$ it holds that $point\_on\_ray(D, A, B)$ or $point\_on\_ray(D, A, C)$ or $D = A$.*

Theorem 67 (th_6_15_2.) *Assuming that $A \neq B$ and $A \neq C$ and $bet(B, A, C)$ and $point\_on\_ray(D, A, B)$ it holds that $point\_on\_line(D, A, B)$.*

   Proof:

1. From the facts $A \neq B$ it holds that $B \neq A$.

2. From the facts $B \neq A$ and $point\_on\_ray(D, A, B)$ it holds that $out(A, D, B)$ (using $ax\_6\_8\_1$).

3. From the facts $out(A, D, B)$ it holds that $col(D, A, B)$ and $not\_bet(D, A, B)$ (using $th\_6\_4\_1$).

4. From the facts $col(D, A, B)$ it holds that $col(A, B, D)$ and $col(B, D, A)$ and $col(B, A, D)$ and $col(A, D, B)$ and $col(D, B, A)$ (using th_4_11).

5. From the facts $A \neq B$ and $col(A, B, D)$ it holds that $point\_on\_line(D, A, B)$ (using $ax\_6\_14\_2$).

6. From the facts $point\_on\_line(D, A, B)$ we proved a conjecture.

QED

Theorem 68 (th_6_15_3.) *Assuming that $A \neq B$ and $A \neq C$ and $bet(B, A, C)$ and $point\_on\_ray(D, A, C)$ it holds that $point\_on\_line(D, A, B)$.*

Theorem 69 (th_6_15_4.) *Assuming that $A \neq B$ and $A \neq C$ and $bet(B, A, C)$ and $D = A$ it holds that $point\_on\_line(D, A, B)$.*

Proof:

1. It holds that $bet(B, A, A)$ (using th_3_1).
2. From the facts $bet(B, A, A)$ it holds that $col(A, B, A)$ (using $ax\_4\_10\_3$).
3. From the facts $A \neq B$ and $col(A, B, A)$ it holds that $point\_on\_line(A, A, B)$ (using $ax\_6\_14\_2$).
4. From the facts $point\_on\_line(A, A, B)$ and $D = A$ it holds that $point\_on\_line(D, D, B)$.
5. From the facts $point\_on\_line(D, D, B)$ and $D = A$ it holds that $point\_on\_line(D, A, B)$.
6. From the facts $point\_on\_line(D, A, B)$ we proved a conjecture.

QED

Theorem 70 (th_6_16.) *Assuming that $A \neq B$ and $C \neq A$ and $point\_on\_line(C, A, B)$ it holds that $same\_lines(A, B, A, C)$.*

Proof:

1. It holds that $bet(B, A, A)$ (using th_3_1).
2. From the facts $bet(B, A, A)$ it holds that $col(A, B, A)$ (using $ax\_4\_10\_3$).
3. From the facts $A \neq B$ and $col(A, B, A)$ it holds that $point\_on\_line(A, A, B)$ (using $ax\_6\_14\_2$).
4. From the facts $C \neq A$ it holds that $A \neq C$.
5. From the facts $A \neq B$ and $A \neq C$ and $point\_on\_line(A, A, B)$ and $point\_on\_line(C, A, B)$ it holds that $same\_lines(A, B, A, C)$ (using $ax\_same\_lines\_1$).
6. From the facts $same\_lines(A, B, A, C)$ we proved a conjecture.

QED

Theorem 71 (th_6_17.) *Assuming that $A \neq B$ it holds that $point\_on\_line(A, A, B)$ and $point\_on\_line(B, A, B)$ and $same\_lines(A, B, B, A)$.*

Proof:

1. It holds that $col(A, A, B)$ (using th_4_12).
2. It holds that $col(B, B, A)$ (using th_4_12).
3. From the facts $col(A, A, B)$ it holds that $col(A, B, A)$ and $col(B, A, A)$ and $col(B, A, A)$ and $col(A, A, B)$ and $col(A, B, A)$ (using th_4_11).
4. From the facts $col(B, B, A)$ it holds that $col(B, A, B)$ and $col(A, B, B)$ and $col(A, B, B)$ and $col(B, B, A)$ and $col(B, A, B)$ (using th_4_11).
5. From the facts $A \neq B$ and $col(A, B, A)$ it holds that $point\_on\_line(A, A, B)$ (using $ax\_6\_14\_2$).
6. From the facts $A \neq B$ and $col(A, B, B)$ it holds that $point\_on\_line(B, A, B)$ (using $ax\_6\_14\_2$).
7. From the facts $A \neq B$ it holds that $B \neq A$.

8. From the facts $A \neq B$ and $B \neq A$ and $point\_on\_line(B, A, B)$ and $point\_on\_line(A, A, B)$ it holds that $same\_lines(A, B, B, A)$ (using $ax\_same\_lines\_1$).

9. From the facts $point\_on\_line(A, A, B)$ and $point\_on\_line(B, A, B)$ and $same\_lines(A, B, B, A)$ we proved a conjecture.

QED

Theorem 72 (th_6_18.) *Assuming that $A \neq B$ and $C \neq D$ and $point\_on\_line(C, A, B)$ and $point\_on\_line(D, A, B)$ it holds that $same\_lines(A, B, C, D)$.*

Proof:

1. From the facts $A \neq B$ and $C \neq D$ and $point\_on\_line(C, A, B)$ and $point\_on\_line(D, A, B)$ it holds that $same\_lines(A, B, C, D)$ (using $ax\_same\_lines\_1$).

2. From the facts $same\_lines(A, B, C, D)$ we proved a conjecture.

QED

Theorem 73 (th_6_19_1.) *Assuming that $A \neq B$ there exist point C, point D, such that $C \neq D$ and $point\_on\_line(A, C, D)$ and $point\_on\_line(B, C, D)$.*

Proof:

1. From the facts $A \neq B$ it holds that $point\_on\_line(A, A, B)$ and $point\_on\_line(B, A, B)$ and $same\_lines(A, B, B, A)$ (using th_6_17).

2. From the facts $A \neq B$ and $point\_on\_line(A, A, B)$ and $point\_on\_line(B, A, B)$ we proved a conjecture.

QED

Theorem 74 (th_6_19_2.) *Assuming that $A \neq B$ and $C \neq D$ and $point\_on\_line(A, C, D)$ and $point\_on\_line(B, C, D)$ and $E \neq F$ and $point\_on\_line(A, E, F)$ and $point\_on\_line(B, E, F)$ it holds that $same\_lines(C, D, E, F)$.*

Theorem 75 (th_6_21.) *Assuming that $A \neq B$ and $C \neq D$ and $not\_same\_lines(A, B, C, D)$ and $point\_on\_line(E, A, B)$ and $point\_on\_line(E, C, D)$ and $point\_on\_line(F, A, B)$ and $point\_on\_line(F, C, D)$ it holds that $E = F$.*

Theorem 76 (th_6_23_1.) *Assuming that $col(A, B, C)$ there exist point D, point E, such that $D \neq E$ and $point\_on\_line(A, D, E)$ and $point\_on\_line(B, D, E)$ and $point\_on\_line(C, D, E)$.*

Proof:

1. It holds that $A = B$ or $A \neq B$ (using ax_g1).

  2. Assume that: $A = B$.

    3. It holds that $A = C$ or $A \neq C$ (using ax_g1).

      4. Assume that: $A = C$.

        5. There exist a point D where $bet(A, A, D)$ and $A \neq D$ (using th_3_14).

        6. From the facts $A \neq D$ it holds that $point\_on\_line(A, A, D)$ and $point\_on\_line(D, A, D)$ and $same\_lines(A, D, D, A)$ (using th_6_17).

7. From the facts $point\_on\_line(A, A, D)$ and $A = B$ it holds that $point\_on\_line(B, A, D)$.

8. From the facts $point\_on\_line(A, A, D)$ and $A = C$ it holds that $point\_on\_line(C, A, D)$.

9. From the facts $A \neq D$ and $point\_on\_line(A, A, D)$ and $point\_on\_line(B, A, D)$ and $point\_on\_line(C, A, D)$ we proved a conjecture.

10. Assume that: $A \neq C$.

11. From the facts $A \neq C$ it holds that $point\_on\_line(A, A, C)$ and $point\_on\_line(C, A, C)$ and $same\_lines(A, C, C, A)$ (using th_6_17).

12. From the facts $point\_on\_line(A, A, C)$ and $A = B$ it holds that $point\_on\_line(B, A, C)$.

13. From the facts $A \neq C$ and $point\_on\_line(A, A, C)$ and $point\_on\_line(B, A, C)$ and $point\_on\_line(C, A, C)$ we proved a conjecture.

14. Assume that: $A \neq B$.

15. From the facts $A \neq B$ it holds that $point\_on\_line(A, A, B)$ and $point\_on\_line(B, A, B)$ and $same\_lines(A, B, B, A)$ (using th_6_17).

16. From the facts $A \neq B$ and $col(A, B, C)$ it holds that $point\_on\_line(C, A, B)$ (using $ax\_6\_14\_2$).

17. From the facts $A \neq B$ and $point\_on\_line(A, A, B)$ and $point\_on\_line(B, A, B)$ and $point\_on\_line(C, A, B)$ we proved a conjecture.

QED

Theorem 77 (th_6_23_2.) *Assuming that $D \neq E$ and $point\_on\_line(A, D, E)$ and $point\_on\_line(B, D, E)$ and $point\_on\_line(C, D, E)$ it holds that $col(A, B, C)$.*

Theorem 78 (th_6_24.) *There exist point A, point B, point C, such that $not\_col(A, B, C)$.*

Theorem 79 (th_6_25.) *Assuming that $A \neq B$ there exist point C, such that $not\_col(A, B, C)$.*

Proof:

1. It holds that $col(A, B, A)$ or $not\_col(A, B, A)$ (using ax_branch_col).

2. Assume that: $col(A, B, A)$.

3. From the facts $A \neq B$ and $col(A, B, A)$ it holds that $point\_on\_line(A, A, B)$ (using $ax\_6\_14\_2$).

4. It holds that $col(A, B, B)$ or $not\_col(A, B, B)$ (using ax_branch_col).

5. Assume that: $col(A, B, B)$.

6. From the facts $A \neq B$ and $col(A, B, B)$ it holds that $point\_on\_line(B, A, B)$ (using $ax\_6\_14\_2$).

7. There exist a point C and a point D and a point E where $not\_col(C, D, E)$ (using th_6_24).

8. It holds that $col(A, B, C)$ or $not\_col(A, B, C)$ (using ax_branch_col).

9. Assume that: $col(A, B, C)$.

10. From the facts $A \neq B$ and $col(A, B, C)$ it holds that $point\_on\_line(C, A, B)$ (using $ax\_6\_14\_2$).

11. It holds that $col(A, B, D)$ or $not\_col(A, B, D)$ (using ax_branch_col).

12. Assume that: $col(A, B, D)$.

13. From the facts $A \neq B$ and $col(A, B, D)$ it holds that $point\_on\_line(D, A, B)$ (using $ax\_6\_14\_2$).

14. It holds that $col(A, B, E)$ or $not\_col(A, B, E)$ (using ax_branch_col).

15. Assume that: $col(A, B, E)$.

16. From the facts $A \neq B$ and $col(A, B, E)$ it holds that $point\_on\_line(E, A, B)$ (using $ax\_6\_14\_2$).

17. From the facts $A \neq B$ and $point\_on\_line(C, A, B)$ and $point\_on\_line(D, A, B)$ and $point\_on\_line(E, A, B)$ it holds that $col(C, D, E)$ (using $th\_6\_23\_2$).

18. From the facts $col(C, D, E)$ and $not\_col(C, D, E)$ we get contradiction (using ax_false_col).

19. Assume that: $not\_col(A, B, E)$.

20. From the facts $not\_col(A, B, E)$ we proved a conjecture.

21. Assume that: $not\_col(A, B, D)$.

22. From the facts $not\_col(A, B, D)$ we proved a conjecture.

23. Assume that: $not\_col(A, B, C)$.

24. From the facts $not\_col(A, B, C)$ we proved a conjecture.

25. Assume that: $not\_col(A, B, B)$.

26. From the facts $not\_col(A, B, B)$ we proved a conjecture.

27. Assume that: $not\_col(A, B, A)$.

28. From the facts $not\_col(A, B, A)$ we proved a conjecture.

QED

Chapter 7

Point reflexivity

Theorem 80 (th_7_2.) *Assuming that is_midpoint(A, B, C) it holds that is_midpoint(A, C, B).*

   Proof:
1. From the facts *is_midpoint*(A, B, C) it holds that $bet(B, A, C)$ and $AB \cong AC$ (using *ax_7_1*).
2. From the facts $bet(B, A, C)$ it holds that $bet(C, A, B)$ (using *th_3_2*).
3. From the facts $AB \cong AC$ it holds that $AC \cong AB$ (using *th_2_2*).
4. From the facts $bet(C, A, B)$ and $AC \cong AB$ it holds that *is_midpoint*(A, C, B) (using *ax_7_2*).
5. From the facts *is_midpoint*(A, C, B) we proved a conjecture.
QED

Theorem 81 (th_7_3_1.) *Assuming that is_midpoint(A, B, B) it holds that $A = B$.*

   Proof:
1. From the facts *is_midpoint*(A, B, B) it holds that $bet(B, A, B)$ and $AB \cong AB$ (using *ax_7_1*).
2. From the facts $bet(B, A, B)$ it holds that $B = A$ (using *ax_6*).
3. From the facts $B = A$ it holds that $A = B$.
4. From the facts $A = B$ we proved a conjecture.
QED

Theorem 82 (th_7_3_2.) *It holds that is_midpoint(A, A, A).*

   Proof:
1. It holds that $bet(A, A, A)$ (using *th_3_1*).
2. It holds that $AA \cong AA$ (using *th_2_1*).
3. From the facts $bet(A, A, A)$ and $AA \cong AA$ it holds that *is_midpoint*(A, A, A) (using *ax_7_2*).

4. From the facts $is\_midpoint(A, A, A)$ we proved a conjecture.

QED

Theorem 83 (th_7_4_1.) *There exist point C, such that $is\_midpoint(A, B, C)$.*

  Proof:

1. There exist a point M where $bet(B, A, M)$ and $AM \cong AB$ (using ax_4).
2. From the facts $AM \cong AB$ it holds that $AB \cong AM$ (using th_2_2).
3. From the facts $bet(B, A, M)$ and $AB \cong AM$ it holds that $is\_midpoint(A, B, M)$ (using ax_7_2).
4. From the facts $is\_midpoint(A, B, M)$ we proved a conjecture.

QED

Theorem 84 (th_7_4_2.) *Assuming that $is\_midpoint(A, B, C)$ and $is\_midpoint(A, B, D)$ it holds that $C = D$.*

  Proof:

1. From the facts $is\_midpoint(A, B, C)$ it holds that $is\_midpoint(A, C, B)$ (using th_7_2).
2. From the facts $is\_midpoint(A, B, D)$ it holds that $is\_midpoint(A, D, B)$ (using th_7_2).
3. From the facts $is\_midpoint(A, B, C)$ it holds that $bet(B, A, C)$ and $AB \cong AC$ (using ax_7_1).
4. From the facts $is\_midpoint(A, B, D)$ it holds that $bet(B, A, D)$ and $AB \cong AD$ (using ax_7_1).
5. From the facts $is\_midpoint(A, C, B)$ it holds that $bet(C, A, B)$ and $AC \cong AB$ (using ax_7_1).
6. From the facts $is\_midpoint(A, D, B)$ it holds that $bet(D, A, B)$ and $AD \cong AB$ (using ax_7_1).
7. It holds that $A = B$ or $A \neq B$ (using ax_g1).
   8. Assume that: $A = B$.
      9. From the facts $AC \cong AB$ and $A = B$ it holds that $AC \cong AA$.
      10. From the facts $AC \cong AA$ it holds that $A = C$ (using ax_3).
      11. From the facts $AD \cong AB$ and $A = B$ and $A = C$ it holds that $AD \cong AA$.
      12. From the facts $AD \cong AA$ it holds that $A = D$ (using ax_3).
      13. From the facts $A = C$ and $A = D$ it holds that $C = D$.
      14. From the facts $C = D$ we proved a conjecture.
   15. Assume that: $A \neq B$.
      16. From the facts $A \neq B$ it holds that $B \neq A$.
      17. From the facts $B \neq A$ and $bet(B, A, C)$ and $AC \cong AB$ and $bet(B, A, D)$ and $AD \cong AB$ it holds that $C = D$ (using th_2_12).
      18. From the facts $C = D$ we proved a conjecture.

QED

Theorem 85 (th_7_7.) *Assuming that is_symmetric(A, B, C) it holds that is_symmetric(B, A, C).*

Proof:

1. From the facts *is_symmetric*(A, B, C) it holds that *is_midpoint*(C, A, B) (using *ax_7_5_1*).

2. From the facts *is_midpoint*(C, A, B) it holds that *is_midpoint*(C, B, A) (using *th_7_2*).

3. From the facts *is_midpoint*(C, B, A) it holds that *is_symmetric*(B, A, C) (using *ax_7_5_2*).

4. From the facts *is_symmetric*(B, A, C) we proved a conjecture.

QED

Theorem 86 (th_7_8_1.) *There exist point C, such that is_symmetric(C, A, B).*

Proof:

1. There exist a point E where *is_midpoint*(B, A, E) (using *th_7_4_1*).

2. From the facts *is_midpoint*(B, A, E) it holds that *is_midpoint*(B, E, A) (using *th_7_2*).

3. From the facts *is_midpoint*(B, E, A) it holds that *is_symmetric*(E, A, B) (using *ax_7_5_2*).

4. From the facts *is_symmetric*(E, A, B) we proved a conjecture.

QED

Theorem 87 (th_7_8_2.) *Assuming that is_symmetric(A, B, C) and is_symmetric(D, B, C) it holds that A = D.*

Proof:

1. From the facts *is_symmetric*(A, B, C) it holds that *is_midpoint*(C, A, B) (using *ax_7_5_1*).

2. From the facts *is_midpoint*(C, A, B) it holds that *is_midpoint*(C, B, A) (using *th_7_2*).

3. From the facts *is_symmetric*(D, B, C) it holds that *is_midpoint*(C, D, B) (using *ax_7_5_1*).

4. From the facts *is_midpoint*(C, D, B) it holds that *is_midpoint*(C, B, D) (using *th_7_2*).

5. From the facts *is_midpoint*(C, B, A) and *is_midpoint*(C, B, D) it holds that A = D (using *th_7_4_2*).

6. From the facts A = D we proved a conjecture.

QED

Theorem 88 (th_7_10_1.) *Assuming that is_symmetric(A, A, B) it holds that A = B.*

Proof:

1. From the facts $is\_symmetric(A, A, B)$ it holds that $is\_midpoint(B, A, A)$ (using $ax\_7\_5\_1$).

2. From the facts $is\_midpoint(B, A, A)$ it holds that $B = A$ (using $th\_7\_3\_1$).

3. From the facts $B = A$ it holds that $A = B$.

4. From the facts $A = B$ we proved a conjecture.

QED

Theorem 89 (th_7_10_2.) *It holds that $is\_symmetric(A, A, A)$.*

   Proof:

1. It holds that $is\_midpoint(A, A, A)$ (using $th\_7\_3\_2$).

2. From the facts $is\_midpoint(A, A, A)$ it holds that $is\_symmetric(A, A, A)$ (using $ax\_7\_5\_2$).

3. From the facts $is\_symmetric(A, A, A)$ we proved a conjecture.

QED

Theorem 90 (th_7_13.) *Assuming that $is\_symmetric(A, D, C)$ and $is\_symmetric(B, E, C)$ it holds that $AB \cong DE$.*

Theorem 91 (th_7_15_1.) *Assuming that $bet(A, B, C)$ and $is\_symmetric(A, D, E)$ and $is\_symmetric(B, F, E)$ and $is\_symmetric(C, G, E)$ it holds that $bet(D, F, G)$.*

   Proof:

1. From the facts $is\_symmetric(A, D, E)$ and $is\_symmetric(B, F, E)$ it holds that $AB \cong DF$ (using th_7_13).

2. From the facts $is\_symmetric(A, D, E)$ and $is\_symmetric(C, G, E)$ it holds that $AC \cong DG$ (using th_7_13).

3. From the facts $is\_symmetric(B, F, E)$ and $is\_symmetric(C, G, E)$ it holds that $BC \cong FG$ (using th_7_13).

4. From the facts $AB \cong DF$ and $AC \cong DG$ and $BC \cong FG$ it holds that $cong3(A, B, C, D, F, G)$ (using $ax\_4\_4\_2$).

5. From the facts $bet(A, B, C)$ and $cong3(A, B, C, D, F, G)$ it holds that $bet(D, F, G)$ (using th_4_6).

6. From the facts $bet(D, F, G)$ we proved a conjecture.

QED

Theorem 92 (th_7_15_2.) *Assuming that $bet(A, B, C)$ and $is\_symmetric(D, A, E)$ and $is\_symmetric(F, B, E)$ and $is\_symmetric(G, C, E)$ it holds that $bet(D, F, G)$.*

   Proof:

1. From the facts $is\_symmetric(D, A, E)$ and $is\_symmetric(F, B, E)$ it holds that $DF \cong AB$ (using th_7_13).

2. From the facts $DF \cong AB$ it holds that $AB \cong DF$ (using th_2_2).

3. From the facts $is\_symmetric(D, A, E)$ and $is\_symmetric(G, C, E)$ it holds that $DG \cong AC$ (using th_7_13).

4. From the facts $DG \cong AC$ it holds that $AC \cong DG$ (using th_2_2).

5. From the facts $is\_symmetric(F, B, E)$ and $is\_symmetric(G, C, E)$ it holds that $FG \cong BC$ (using th_7_13).

6. From the facts $FG \cong BC$ it holds that $BC \cong FG$ (using th_2_2).

7. From the facts $AB \cong DF$ and $AC \cong DG$ and $BC \cong FG$ it holds that $cong3(A, B, C, D, F, G)$ (using $ax\_4\_4\_2$).

8. From the facts $bet(A, B, C)$ and $cong3(A, B, C, D, F, G)$ it holds that $bet(D, F, G)$ (using th_4_6).

9. From the facts $bet(D, F, G)$ we proved a conjecture.

QED

Theorem 93 (th_7_16_1.) *Assuming that $AB \cong CD$ and $is\_symmetric(A, E, F)$ and $is\_symmetric(B, G, F)$ and $is\_symmetric(C, I, F)$ and $is\_symmetric(D, J, F)$ it holds that $EG \cong IJ$.*

Proof:

1. From the facts $is\_symmetric(A, E, F)$ and $is\_symmetric(B, G, F)$ it holds that $AB \cong EG$ (using th_7_13).

2. From the facts $is\_symmetric(C, I, F)$ and $is\_symmetric(D, J, F)$ it holds that $CD \cong IJ$ (using th_7_13).

3. From the facts $AB \cong CD$ and $AB \cong EG$ it holds that $CD \cong EG$ (using ax_2).

4. From the facts $CD \cong EG$ and $CD \cong IJ$ it holds that $EG \cong IJ$ (using ax_2).

5. From the facts $EG \cong IJ$ we proved a conjecture.

QED

Theorem 94 (th_7_16_2.) *Assuming that $AB \cong CD$ and $is\_symmetric(E, A, F)$ and $is\_symmetric(G, B, F)$ and $is\_symmetric(I, C, F)$ and $is\_symmetric(J, D, F)$ it holds that $EG \cong IJ$.*

Proof:

1. From the facts $is\_symmetric(E, A, F)$ it holds that $is\_symmetric(A, E, F)$ (using th_7_7).

2. From the facts $is\_symmetric(G, B, F)$ it holds that $is\_symmetric(B, G, F)$ (using th_7_7).

3. From the facts $is\_symmetric(I, C, F)$ it holds that $is\_symmetric(C, I, F)$ (using th_7_7).

4. From the facts $is\_symmetric(J, D, F)$ it holds that $is\_symmetric(D, J, F)$ (using th_7_7).

5. From the facts $AB \cong CD$ and $is\_symmetric(A, E, F)$ and $is\_symmetric(B, G, F)$ and $is\_symmetric(C, I, F)$ and $is\_symmetric(D, J, F)$ it holds that $EG \cong IJ$ (using $th\_7\_16\_1$).

6. From the facts $EG \cong IJ$ we proved a conjecture.

QED
_____

Theorem 95 (th_7_17.) *Assuming that is_midpoint(A, B, C) and is_midpoint(D, B, C) it holds that A = D.*

    Proof:

1. It holds that $bet(A, B, B)$ (using th_3_1).

2. It holds that $bet(D, A, A)$ (using th_3_1).

3. From the facts $is\_midpoint(A, B, C)$ it holds that $bet(B, A, C)$ and $AB \cong AC$ (using ax_7_1).

4. From the facts $is\_midpoint(D, B, C)$ it holds that $bet(B, D, C)$ and $DB \cong DC$ (using ax_7_1).

5. From the facts $bet(B, A, C)$ it holds that $col(C, B, A)$ (using $ax\_4\_10\_3$).

6. From the facts $bet(B, D, C)$ it holds that $col(C, B, D)$ (using $ax\_4\_10\_3$).

7. From the facts $bet(D, A, A)$ it holds that $col(A, D, A)$ (using $ax\_4\_10\_3$).

8. It holds that $A = B$ or $A \neq B$ (using ax_g1).

  9. Assume that: $A = B$.

   10. It holds that $A = C$ or $A \neq C$ (using ax_g1).

    11. Assume that: $A = C$.

     12. From the facts $bet(B, D, C)$ and $A = B$ and $A = C$ it holds that $bet(A, D, A)$.

     13. From the facts $bet(A, D, A)$ and $bet(D, A, A)$ it holds that $A = D$ (using th_3_4).

     14. From the facts $A = D$ it holds that $A = D$.

     15. From the facts $A = D$ we proved a conjecture.

    16. Assume that: $A \neq C$.

     17. From the facts $col(C, B, A)$ and $A = B$ it holds that $col(C, A, A)$.

     18. From the facts $col(C, B, D)$ and $A = B$ it holds that $col(C, A, D)$.

     19. It holds that $A = D$ or $A \neq D$ (using ax_g1).

      20. Assume that: $A = D$.

       21. From the facts $A = D$ it holds that $A = D$.

       22. From the facts $A = D$ we proved a conjecture.

      23. Assume that: $A \neq D$.

       24. From the facts $AB \cong AC$ and $A = B$ it holds that $AA \cong AC$.

       25. From the facts $DB \cong DC$ and $A = B$ it holds that $DA \cong DC$.

       26. From the facts $A \neq D$ and $col(A, D, A)$ and $AA \cong AC$ and $DA \cong DC$ it holds that $A = C$ (using th_4_18).

       27. From the facts $A = C$ and $A \neq C$ we get contradiction.

  28. Assume that: $A \neq B$.

   29. From the facts $A \neq B$ it holds that $point\_on\_line(A, A, B)$ and $point\_on\_line(B, A, B)$ and $same\_lines(A, B, B, A)$ (using th_6_17).

   30. It holds that $A = D$ or $A \neq D$ (using ax_g1).

31. Assume that: $A = D$.

    32. From the facts $A = D$ we proved a conjecture.

33. Assume that: $A \neq D$.

    34. It holds that $B = C$ or $B \neq C$ (using ax_g1).

        35. Assume that: $B = C$.

          36. From the facts $bet(B, A, C)$ and $B = C$ it holds that $bet(B, A, B)$.

          37. From the facts $bet(A, B, B)$ and $bet(B, A, B)$ it holds that $A = B$ (using th_3_4).

          38. From the facts $A = B$ and $A \neq B$ we get contradiction.

        39. Assume that: $B \neq C$.

          40. From the facts $B \neq C$ it holds that $C \neq B$.

          41. From the facts $C \neq B$ and $col(C, B, A)$ it holds that $point\_on\_line(A, C, B)$ (using $ax\_6\_14\_2$).

          42. From the facts $B \neq C$ it holds that $C \neq B$.

          43. From the facts $C \neq B$ and $col(C, B, D)$ it holds that $point\_on\_line(D, C, B)$ (using $ax\_6\_14\_2$).

          44. From the facts $B \neq C$ it holds that $C \neq B$.

          45. From the facts $C \neq B$ it holds that $point\_on\_line(C, C, B)$ and $point\_on\_line(B, C, B)$ and $same\_lines(C, B, B, C)$ (using th_6_17).

          46. From the facts $B \neq C$ it holds that $C \neq B$.

          47. From the facts $C \neq B$ and $point\_on\_line(A, C, B)$ and $point\_on\_line(B, C, B)$ and $point\_on\_line(D, C, B)$ it holds that $col(A, B, D)$ (using $th\_6\_23\_2$).

          48. From the facts $A \neq B$ and $col(A, B, D)$ it holds that $point\_on\_line(D, A, B)$ (using $ax\_6\_14\_2$).

          49. From the facts $A \neq B$ and $point\_on\_line(A, A, B)$ and $point\_on\_line(D, A, B)$ and $point\_on\_line(B, A, B)$ it holds that $col(A, D, B)$ (using $th\_6\_23\_2$).

          50. From the facts $A \neq D$ and $col(A, D, B)$ and $AB \cong AC$ and $DB \cong DC$ it holds that $B = C$ (using th_4_18).

          51. From the facts $B = C$ and $B \neq C$ we get contradiction.

QED

Theorem 96 (th_7_18.) *Assuming that is_symmetric(A, B, C) and is_symmetric(A, B, D) it holds that $C = D$.*

    Proof:

1. From the facts *is_symmetric*(A, B, C) it holds that *is_midpoint*(C, A, B) (using $ax\_7\_5\_1$).

2. From the facts *is_symmetric*(A, B, D) it holds that *is_midpoint*(D, A, B) (using $ax\_7\_5\_1$).

3. From the facts *is_midpoint*(C, A, B) and *is_midpoint*(D, A, B) it holds that $C = D$ (using th_7_17).

4. From the facts $C = D$ we proved a conjecture.

QED

Theorem 97 (th_7_19_1.) *Assuming that is_symmetric(C, D, B) and is_symmetric(D, F, A) and is_symmetric(C, E, A) and is_symmetric(E, F, B) it holds that $A = B$.*

Theorem 98 (th_7_19_2.) *Assuming that is_symmetric(A, B, C) and is_symmetric(B, D, E) and is_symmetric(A, F, E) and is_symmetric(F, G, C) and $E = C$ it holds that $D = G$.*

   Proof:
1. From the facts *is_symmetric*(A, F, E) and $E = C$ it holds that *is_symmetric*(A, F, C).
2. From the facts *is_symmetric*(B, D, E) and $E = C$ it holds that *is_symmetric*(B, D, C).
3. From the facts *is_symmetric*(B, D, C) it holds that *is_symmetric*(D, B, C) (using *th_7_7*).
4. From the facts *is_symmetric*(F, G, C) it holds that *is_symmetric*(G, F, C) (using *th_7_7*).
5. From the facts *is_symmetric*(A, B, C) and *is_symmetric*(D, B, C) it holds that $A = D$ (using *th_7_8_2*).
6. From the facts *is_symmetric*(A, F, E) and $E = C$ it holds that *is_symmetric*(A, F, C).
7. From the facts *is_symmetric*(A, F, C) and *is_symmetric*(G, F, C) it holds that $A = G$ (using *th_7_8_2*).
8. From the facts $A = D$ and $A = G$ it holds that $D = G$.
9. From the facts $D = G$ we proved a conjecture.
QED

Theorem 99 (th_7_20.) *Assuming that col(A, B, C) and $BA \cong BC$ it holds that $A = C$ or is_midpoint(B, A, C).*

Theorem 100 (th_7_21.) *Assuming that not_col(A, B, C) and $B \neq D$ and $AB \cong CD$ and $BC \cong DA$ and col(A, E, C) and col(B, E, D) it holds that is_midpoint(E, A, C) and is_midpoint(E, B, D).*

Theorem 101 (th_7_22.) *Assuming that bet(A, E, B) and bet(C, E, D) and $EA \cong EC$ and $EB \cong ED$ and is_midpoint(F, A, C) and is_midpoint(G, B, D) it holds that bet(F, E, G).*

Theorem 102 (th_7_25.) *Assuming that $CA \cong CB$ there exist point D, such that is_midpoint(D, A, B).*

Chapter 8

Right Angles

Theorem 103 (th_8_2.) *Assuming that $per(A, B, C)$ it holds that $per(C, B, A)$.*

Theorem 104 (th_8_3.) *Assuming that $per(A, B, C)$ and $A \neq B$ and $col(B, A, D)$ it holds that $per(D, B, C)$.*

Theorem 105 (th_8_4.) *Assuming that $per(A, B, C)$ and $is\_midpoint(B, C, D)$ it holds that $per(A, B, D)$.*

    Proof:

1. From the facts $is\_midpoint(B, C, D)$ it holds that $is\_midpoint(B, D, C)$ (using th_7_2).

2. From the facts $per(A, B, C)$ there exist a point E where $AC \cong AE$ and $is\_midpoint(B, C, E)$ (using $ax\_8\_1\_1$).

3. From the facts $is\_midpoint(B, C, D)$ and $is\_midpoint(B, C, E)$ it holds that $D = E$ (using $th\_7\_4\_2$).

4. From the facts $AC \cong AE$ and $D = E$ it holds that $AC \cong AD$.

5. From the facts $AC \cong AD$ it holds that $AD \cong AC$ (using th_2_2).

6. From the facts $AD \cong AC$ and $is\_midpoint(B, D, C)$ it holds that $per(A, B, D)$ (using $ax\_8\_1\_2$).

7. From the facts $per(A, B, D)$ we proved a conjecture.

QED

Theorem 106 (th_8_5.) *It holds that $per(A, B, B)$.*

    Proof:

1. It holds that $is\_midpoint(B, B, B)$ (using $th\_7\_3\_2$).

2. It holds that $AB \cong AB$ (using th_2_1).

3. From the facts $AB \cong AB$ and $is\_midpoint(B, B, B)$ it holds that $per(A, B, B)$ (using $ax\_8\_1\_2$).

4. From the facts $per(A, B, B)$ we proved a conjecture.
QED

Theorem 107 (th_8_6.) *Assuming that $per(A, B, C)$ and $per(D, B, C)$ and $bet(A, C, D)$ it holds that $B = C$.*

Theorem 108 (th_8_7.) *Assuming that $per(A, B, C)$ and $per(A, C, B)$ it holds that $B = C$.*

Theorem 109 (th_8_8.) *Assuming that $per(A, B, A)$ it holds that $A = B$.*

   Proof:
1. It holds that $bet(A, A, A)$ (using th_3_1).
2. From the facts $per(A, B, A)$ and $per(A, B, A)$ and $bet(A, A, A)$ it holds that $B = A$ (using th_8_6).
3. From the facts $B = A$ it holds that $A = B$.
4. From the facts $A = B$ we proved a conjecture.
QED

Theorem 110 (th_8_9.) *Assuming that $per(A, B, C)$ and $col(A, B, C)$ it holds that $A = B$ or $C = B$.*

   Proof:
1. From the facts $col(A, B, C)$ it holds that $col(B, C, A)$ and $col(C, A, B)$ and $col(C, B, A)$ and $col(B, A, C)$ and $col(A, C, B)$ (using th_4_11).
2. It holds that $A = B$ or $A \neq B$ (using ax_g1).
   3. Assume that: $A = B$.
      4. From the facts $A = B$ we proved a conjecture.
   5. Assume that: $A \neq B$.
      6. From the facts $per(A, B, C)$ and $A \neq B$ and $col(B, A, C)$ it holds that $per(C, B, C)$ (using th_8_3).
      7. From the facts $per(C, B, C)$ it holds that $C = B$ (using th_8_8).
      8. From the facts $C = B$ we proved a conjecture.
QED

Theorem 111 (th_8_10.) *Assuming that $per(A, B, C)$ and $cong3(A, B, C, D, E, F)$ it holds that $per(D, E, F)$.*

Theorem 112 (th_8_12.) *Assuming that $perp\_in(A, B, C, D, E)$ it holds that $perp\_in(A, D, E, B, C)$.*

Theorem 113 (th_8_13_1.) *Assuming that $perp\_in(A, B, C, D, E)$ there exist point F, point G, such that $B \neq C$ and $D \neq E$ and $point\_on\_line(A, B, C)$ and $point\_on\_line(A, D, E)$ and $point\_on\_line(F, B, C)$ and $point\_on\_line(G, D, E)$ and $F \neq A$ and $G \neq A$ and $per(F, A, G)$.*

Theorem 114 (th_8_13_2.) *Assuming that $B \neq C$ and $D \neq E$ and $point\_on\_line(A, B, C)$ and $point\_on\_line(A, D, E)$ and $point\_on\_line(F, B, C)$ and $point\_on\_line(G, D, E)$ and $F \neq A$ and $G \neq A$ and $per(F, A, G)$ it holds that $perp\_in(A, B, C, D, E)$.*

Theorem 115 (th_8_14_1.) *Assuming that $perp(A, B, C, D)$ it holds that $A \neq B$ and $C \neq D$ and $not\_same\_lines(A, B, C, D)$.*

Theorem 116 (th_8_14_2_1.) *Assuming that $perp\_in(A, B, C, D, E)$ it holds that $perp(B, C, D, E)$ and $inter(A, B, C, D, E)$.*

Theorem 117 (th_8_14_2_2.) *Assuming that $perp(B, C, D, E)$ and $inter(A, B, C, D, E)$ it holds that $perp\_in(A, B, C, D, E)$.*

Theorem 118 (th_8_14_3.) *Assuming that $perp\_in(E, A, B, C, D)$ and $perp\_in(F, A, B, C, D)$ it holds that $E = F$.*

Theorem 119 (th_8_15_1.) *Assuming that $A \neq B$ and $col(A, B, C)$ and $perp(A, B, D, C)$ it holds that $perp\_in(C, A, B, D, C)$.*

Proof:
1. From the facts $perp(A, B, D, C)$ it holds that $A \neq B$ and $D \neq C$ and $not\_same\_lines(A, B, D, C)$ (using $th\_8\_14\_1$).
2. From the facts $D \neq C$ it holds that $point\_on\_line(D, D, C)$ and $point\_on\_line(C, D, C)$ and $same\_lines(D, C, C, D)$ (using th_6_17).
3. From the facts $A \neq B$ and $col(A, B, C)$ it holds that $point\_on\_line(C, A, B)$ (using $ax\_6\_14\_2$).
4. From the facts $A \neq B$ and $D \neq C$ and $point\_on\_line(C, A, B)$ and $point\_on\_line(C, D, C)$ and $not\_same\_lines(A, B, D, C)$ it holds that $inter(C, A, B, D, C)$ (using $ax\_6\_22\_2$).
5. From the facts $perp(A, B, D, C)$ and $inter(C, A, B, D, C)$ it holds that $perp\_in(C, A, B, D, C)$ (using $th\_8\_14\_2\_2$).
6. From the facts $perp\_in(C, A, B, D, C)$ we proved a conjecture.
QED

Theorem 120 (th_8_15_2.) *Assuming that $A \neq B$ and $col(A, B, C)$ and $perp\_in(C, A, B, D, C)$ it holds that $perp(A, B, D, C)$.*

Proof:
1. From the facts $perp\_in(C, A, B, D, C)$ it holds that $perp(A, B, D, C)$ (using $ax\_8\_11\_2\_2$).
2. From the facts $perp(A, B, D, C)$ we proved a conjecture.
QED

Theorem 121 (th_8_16_1.) *Assuming that $A \neq B$ and $col(A, B, D)$ and $col(A, B, E)$ and $E \neq D$ and $perp(A, B, C, D)$ it holds that $not\_col(A, B, C)$ and $per(C, D, E)$.*

Theorem 122 (th_8_16_2.) *Assuming that $A \neq B$ and $col(A, B, D)$ and $col(A, B, E)$ and $E \neq D$ and $not\_col(A, B, C)$ and $per(C, D, E)$ it holds that $perp(A, B, C, D)$.*

Theorem 123 (th_8_18_1.) *Assuming that $not\_col(A, B, C)$ there exist point D, such that $col(A, B, D)$ and $perp(A, B, C, D)$.*

Theorem 124 (th_8_18_2.) *Assuming that $not\_col(A, B, C)$ and $col(A, B, D)$ and $perp(A, B, C, D)$ and $col(A, B, E)$ and $perp(A, B, C, E)$ it holds that $D = E$.*

    Proof:

1. From the facts $perp(A, B, C, D)$ it holds that $A \neq B$ and $C \neq D$ and $not\_same\_lines(A, B, C, D)$ (using $th\_8\_14\_1$).

2. From the facts $perp(A, B, C, E)$ it holds that $A \neq B$ and $C \neq E$ and $not\_same\_lines(A, B, C, E)$ (using $th\_8\_14\_1$).

3. It holds that $A = D$ or $A \neq D$ (using ax_g1).

    4. Assume that: $A = D$.

        5. It holds that $A = E$ or $A \neq E$ (using ax_g1).

            6. Assume that: $A = E$.

                7. From the facts $A = D$ and $A = E$ it holds that $D = E$.

                8. From the facts $D = E$ we proved a conjecture.

            9. Assume that: $A \neq E$.

          10. From the facts $col(A, B, D)$ and $A = D$ it holds that $col(A, B, A)$.

          11. From the facts $A \neq E$ it holds that $E \neq A$.

          12. From the facts $perp(A, B, C, D)$ and $A = D$ it holds that $perp(A, B, C, A)$.

          13. From the facts $A \neq B$ and $col(A, B, A)$ and $col(A, B, E)$ and $E \neq A$ and $perp(A, B, C, A)$ it holds that $not\_col(A, B, C)$ and $per(C, A, E)$ (using $th\_8\_16\_1$).

          14. From the facts $col(A, B, D)$ and $A = D$ it holds that $col(A, B, A)$.

          15. From the facts $A \neq B$ and $col(A, B, E)$ and $col(A, B, A)$ and $A \neq E$ and $perp(A, B, C, E)$ it holds that $not\_col(A, B, C)$ and $per(C, E, A)$ (using $th\_8\_16\_1$).

          16. From the facts $per(C, A, E)$ and $per(C, E, A)$ it holds that $A = E$ (using th_8_7).

          17. From the facts $A = E$ and $A \neq E$ we get contradiction.

    18. Assume that: $A \neq D$.

        19. It holds that $B = D$ or $B \neq D$ (using ax_g1).

            20. Assume that: $B = D$.

                21. It holds that $B = E$ or $B \neq E$ (using ax_g1).

22. Assume that: $B = E$.

    23. From the facts $B = D$ and $B = E$ it holds that $D = E$.

    24. From the facts $D = E$ we proved a conjecture.

25. Assume that: $B \neq E$.

    26. From the facts $col(A, B, D)$ and $B = D$ it holds that $col(A, B, B)$.

    27. From the facts $B \neq E$ it holds that $E \neq B$.

    28. From the facts $perp(A, B, C, D)$ and $B = D$ it holds that $perp(A, B, C, B)$.

    29. From the facts $A \neq B$ and $col(A, B, B)$ and $col(A, B, E)$ and $E \neq B$ and $perp(A, B, C, B)$ it holds that $not\_col(A, B, C)$ and $per(C, B, E)$ (using $th\_8\_16\_1$).

    30. From the facts $col(A, B, D)$ and $B = D$ it holds that $col(A, B, B)$.

    31. From the facts $A \neq B$ and $col(A, B, E)$ and $col(A, B, B)$ and $B \neq E$ and $perp(A, B, C, E)$ it holds that $not\_col(A, B, C)$ and $per(C, E, B)$ (using $th\_8\_16\_1$).

    32. From the facts $per(C, B, E)$ and $per(C, E, B)$ it holds that $B = E$ (using th_8_7).

    33. From the facts $B = E$ and $B \neq E$ we get contradiction.

34. Assume that: $B \neq D$.

    35. It holds that $D = E$ or $D \neq E$ (using ax_g1).

    36. Assume that: $D = E$.

        37. From the facts $D = E$ we proved a conjecture.

    38. Assume that: $D \neq E$.

        39. From the facts $D \neq E$ it holds that $E \neq D$.

        40. From the facts $A \neq B$ and $col(A, B, D)$ and $col(A, B, E)$ and $E \neq D$ and $perp(A, B, C, D)$ it holds that $not\_col(A, B, C)$ and $per(C, D, E)$ (using $th\_8\_16\_1$).

        41. From the facts $A \neq B$ and $col(A, B, E)$ and $col(A, B, D)$ and $D \neq E$ and $perp(A, B, C, E)$ it holds that $not\_col(A, B, C)$ and $per(C, E, D)$ (using $th\_8\_16\_1$).

        42. From the facts $per(C, D, E)$ and $per(C, E, D)$ it holds that $D = E$ (using th_8_7).

        43. From the facts $D = E$ and $D \neq E$ we get contradiction.

QED

Theorem 125 (th_8_20_1.) *Assuming that $per(A, B, C)$ and $is\_symmetric(C, E, A)$ and $is\_symmetric(C, F, B)$ and $is\_midpoint(D, E, F)$ it holds that $per(B, A, D)$.*

Theorem 126 (th_8_20_2.) *Assuming that $per(A, B, C)$ and $is\_symmetric(C, D, A)$ and $is\_symmetric(C, E, B)$ and $is\_midpoint(F, D, E)$ and $B \neq C$ it holds that $A \neq F$.*

    Proof:

1. From the facts $is\_symmetric(C, D, A)$ it holds that $is\_midpoint(A, C, D)$ (using $ax\_7\_5\_1$).

2. From the facts $is\_midpoint(A, C, D)$ it holds that $is\_midpoint(A, D, C)$ (using th_7_2).

3. From the facts $is\_symmetric(C, E, B)$ it holds that $is\_midpoint(B, C, E)$ (using $ax\_7\_5\_1$).

4. It holds that $bet(B, C, C)$ (using th_3_1).

5. It holds that $bet(F, D, D)$ (using th_3_1).

6. From the facts $is\_midpoint(B, C, E)$ it holds that $bet(C, B, E)$ and $BC \cong BE$ (using ax_7_1).

7. From the facts $is\_midpoint(F, D, E)$ it holds that $bet(D, F, E)$ and $FD \cong FE$ (using ax_7_1).

8. It holds that $A = B$ or $A \neq B$ (using ax_g1).

  9. Assume that: $A = B$.

   10. From the facts $is\_midpoint(B, C, E)$ and $A = B$ it holds that $is\_midpoint(A, C, E)$.

   11. From the facts $is\_midpoint(A, C, D)$ and $is\_midpoint(A, C, E)$ it holds that $D = E$ (using $th\_7\_4\_2$).

   12. From the facts $bet(D, F, E)$ and $D = E$ it holds that $bet(D, F, D)$.

   13. From the facts $bet(D, F, D)$ and $bet(F, D, D)$ it holds that $D = F$ (using th_3_4).

   14. It holds that $A = D$ or $A \neq D$ (using ax_g1).

    15. Assume that: $A = D$.

     16. From the facts $is\_midpoint(F, D, E)$ and $A = B$ and $D = E$ and $D = F$ and $A = D$ it holds that $is\_midpoint(A, A, A)$.

     17. From the facts $is\_midpoint(A, D, C)$ and $A = B$ and $D = E$ and $D = F$ and $A = D$ it holds that $is\_midpoint(A, A, C)$.

     18. From the facts $is\_midpoint(A, A, A)$ and $is\_midpoint(A, A, C)$ it holds that $A = C$ (using $th\_7\_4\_2$).

     19. From the facts $B \neq C$ and $A = B$ and $D = E$ and $D = F$ and $A = D$ it holds that $A \neq C$.

     20. From the facts $B \neq C$ and $A = B$ it holds that $A \neq C$.

     21. From the facts $A \neq C$ it holds that $A \neq C$.

     22. From the facts $A = C$ and $A \neq C$ we get contradiction.

    23. Assume that: $A \neq D$.

     24. From the facts $A \neq D$ it holds that $A \neq D$.

     25. From the facts $A \neq D$ and $D = F$ it holds that $A \neq F$.

     26. From the facts $A \neq D$ and $D = F$ it holds that $A \neq F$.

     27. From the facts $A \neq F$ we proved a conjecture.

  28. Assume that: $A \neq B$.

   29. It holds that $A = F$ or $A \neq F$ (using ax_g1).

    30. Assume that: $A = F$.

31. From the facts $is\_midpoint(F, D, E)$ and $A = F$ it holds that $is\_midpoint(A, D, E)$.
32. From the facts $is\_midpoint(A, D, C)$ and $is\_midpoint(A, D, E)$ it holds that $C = E$ (using $th\_7\_4\_2$).
33. From the facts $bet(C, B, E)$ and $C = E$ it holds that $bet(C, B, C)$.
34. From the facts $bet(B, C, C)$ and $bet(C, B, C)$ it holds that $B = C$ (using th_3_4).
35. From the facts $B = C$ and $B \neq C$ we get contradiction.
36. Assume that: $A \neq F$.
37. From the facts $A \neq F$ we proved a conjecture.

QED

Theorem 127 (th_8_21.) *Assuming that $A \neq B$ there exist point D, point E, such that $perp(A, B, D, A)$ and $col(A, B, E)$ and $bet(C, E, D)$.*

Theorem 128 (th_8_22_1.) *There exist point C, such that $is\_midpoint(C, A, B)$.*

Theorem 129 (th_8_22_2.) *Assuming that $is\_midpoint(A, B, C)$ and $is\_midpoint(D, B, C)$ it holds that $A = D$.*

    Proof:
1. From the facts $is\_midpoint(A, B, C)$ and $is\_midpoint(D, B, C)$ it holds that $A = D$ (using th_7_17).
2. From the facts $A = D$ we proved a conjecture.

QED

Theorem 130 (th_8_24.) *Assuming that $perp(C, A, A, B)$ and $perp(D, B, A, B)$ and $col(A, B, E)$ and $bet(C, E, D)$ and $bet(B, F, D)$ and $AC \cong BF$ there exist point G, such that $is\_midpoint(G, A, B)$ and $is\_midpoint(G, C, F)$.*

Chapter 9

Half-planes and Planes, Subspaces

Theorem 131 (th_9_2.) *Assuming that $two\_sides(A, B, C, D)$ it holds that $two\_sides(B, A, C, D)$.*

Proof:

1. From the facts $two\_sides(A, B, C, D)$ there exist a point E where $C \neq D$ and $not\_point\_on\_line(A, C, D)$ and $not\_point\_on\_line(B, C, D)$ and $point\_on\_line(E, C, D)$ and $bet(A, E, B)$ (using $ax\_9\_1\_1$).

2. From the facts $bet(A, E, B)$ it holds that $bet(B, E, A)$ (using th_3_2).

3. From the facts $C \neq D$ and $not\_point\_on\_line(B, C, D)$ and $not\_point\_on\_line(A, C, D)$ and $point\_on\_line(E, C, D)$ and $bet(B, E, A)$ it holds that $two\_sides(B, A, C, D)$ (using $ax\_9\_1\_2$).

4. From the facts $two\_sides(B, A, C, D)$ we proved a conjecture.

QED

Theorem 132 (th_9_3.) *Assuming that $two\_sides(C, E, A, B)$ and $A \neq B$ and $point\_on\_line(F, A, B)$ and $is\_midpoint(F, C, E)$ and $point\_on\_line(G, A, B)$ and $out(G, C, D)$ it holds that $two\_sides(D, E, A, B)$.*

Theorem 133 (th_9_4_1.) *Assuming that $two\_sides(C, D, A, B)$ and $A \neq B$ and $point\_on\_line(E, A, B)$ and $perp(A, B, C, E)$ and $point\_on\_line(F, A, B)$ and $perp(A, B, D, F)$ and $is\_midpoint(G, E, F)$ and $out(E, I, C)$ and $is\_symmetric(I, J, G)$ it holds that $out(F, J, D)$.*

Theorem 134 (th_9_4_2.) *Assuming that $two\_sides(C, D, A, B)$ and $A \neq B$ and $point\_on\_line(E, A, B)$ and $perp(A, B, C, E)$ and $point\_on\_line(F, A, B)$ and $perp(A, B, D, F)$ and $is\_midpoint(G, E, F)$ and $out(F, J, D)$ and $is\_symmetric(I, J, G)$ it holds that $out(E, I, C)$.*

Theorem 135 (th_9_4_3.) *Assuming that* $two\_sides(C, D, A, B)$ *and* $A \neq B$ *and* $point\_on\_line(E, A, B)$ *and* $perp(A, B, C, E)$ *and* $point\_on\_line(F, A, B)$ *and* $perp(A, B, D, F)$ *and* $out(E, G, C)$ *and* $out(F, I, D)$ *it holds that* $two\_sides(G, I, A, B)$.

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Theorem 136 (th_9_5.) *Assuming that* $A \neq B$ *and* $two\_sides(C, E, A, B)$ *and* $point\_on\_line(F, A, B)$ *and* $out(F, C, D)$ *it holds that* $two\_sides(D, E, A, B)$.

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Theorem 137 (th_9_6.) *Assuming that* $bet(A, C, D)$ *and* $bet(B, E, C)$ *there exist point F, such that* $bet(A, F, B)$ *and* $bet(D, E, F)$.

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Theorem 138 (th_9_8_1.) *Assuming that* $two\_sides(A, B, C, D)$ *and* $two\_sides(E, B, C, D)$ *it holds that* $one\_side(A, E, C, D)$.

   Proof:

1. It holds that $C = D$ or $C \neq D$ (using ax_g1).
   2. Assume that: $C = D$.
      3. From the facts $two\_sides(A, B, C, D)$ and $C = D$ it holds that $two\_sides(A, B, C, C)$.
      4. From the facts $two\_sides(A, B, C, C)$ there exist a point F where $C \neq C$ and $not\_point\_on\_line(A, C, C)$ and $not\_point\_on\_line(B, C, C)$ and $point\_on\_line(F, C, C)$ and $bet(A, F, B)$ (using $ax\_9\_1\_1$).
      5. From the facts $C \neq C$ we get contradiction.
   6. Assume that: $C \neq D$.
      7. From the facts $C \neq D$ and $two\_sides(A, B, C, D)$ and $two\_sides(E, B, C, D)$ it holds that $one\_side(A, E, C, D)$ (using $ax\_9\_7\_2$).
      8. From the facts $one\_side(A, E, C, D)$ we proved a conjecture.
QED ―――――――――――――――――――――――――――――――――――

Theorem 139 (th_9_8_2.) *Assuming that* $two\_sides(C, E, A, B)$ *and* $one\_side(C, D, A, B)$ *it holds that* $two\_sides(D, E, A, B)$.

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Theorem 140 (th_9_9.) *Assuming that* $two\_sides(C, D, A, B)$ *it holds that* $not\_one\_side(C, D, A, B)$.

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Theorem 141 (th_9_10.) *Assuming that* $A \neq B$ *and* $not\_point\_on\_line(C, A, B)$ *there exist point D, such that* $two\_sides(C, D, A, B)$.

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Theorem 142 (th_9_11.) *Assuming that* $A \neq B$ *and* $not\_point\_on\_line(C, A, B)$ *it holds that* $one\_side(C, C, A, B)$.

   Proof:

1. From the facts $A \neq B$ and $not\_point\_on\_line(C, A, B)$ there exist a point D where $two\_sides(C, D, A, B)$ (using th_9_10).

2. From the facts $two\_sides(C, D, A, B)$ and $two\_sides(C, D, A, B)$ it holds that $one\_side(C, C, A, B)$ (using $th\_9\_8\_1$).

3. From the facts $one\_side(C, C, A, B)$ we proved a conjecture.

QED

Theorem 143 (th_9_12.) *Assuming that $one\_side(A, B, C, D)$ it holds that $one\_side(B, A, C, D)$.*

Proof:

1. It holds that $A = B$ or $A \neq B$ (using ax_g1).

  2. Assume that: $A = B$.

    3. From the facts $one\_side(A, B, C, D)$ and $A = B$ it holds that $one\_side(A, A, C, D)$.

    4. From the facts $one\_side(A, A, C, D)$ and $A = B$ it holds that $one\_side(B, A, C, D)$.

    5. From the facts $one\_side(B, A, C, D)$ we proved a conjecture.

  6. Assume that: $A \neq B$.

    7. From the facts $one\_side(A, B, C, D)$ there exist a point E where $C \neq D$ and $two\_sides(A, E, C, D)$ and $two\_sides(B, E, C, D)$ (using $ax\_9\_7\_1$).

    8. From the facts $two\_sides(B, E, C, D)$ and $two\_sides(A, E, C, D)$ it holds that $one\_side(B, A, C, D)$ (using $th\_9\_8\_1$).

    9. From the facts $one\_side(B, A, C, D)$ we proved a conjecture.

QED

Theorem 144 (th_9_13.) *Assuming that $one\_side(A, B, C, D)$ and $one\_side(B, E, C, D)$ it holds that $one\_side(A, E, C, D)$.*

Proof:

1. It holds that $A = B$ or $A \neq B$ (using ax_g1).

  2. Assume that: $A = B$.

    3. From the facts $one\_side(B, E, C, D)$ and $A = B$ it holds that $one\_side(A, E, C, D)$.

    4. From the facts $one\_side(A, E, C, D)$ we proved a conjecture.

  5. Assume that: $A \neq B$.

    6. It holds that $C = D$ or $C \neq D$ (using ax_g1).

      7. Assume that: $C = D$.

      8. From the facts $one\_side(A, B, C, D)$ and $C = D$ it holds that $one\_side(A, B, C, C)$.

      9. From the facts $one\_side(A, B, C, C)$ there exist a point F where $C \neq C$ and $two\_sides(A, F, C, C)$ and $two\_sides(B, F, C, C)$ (using $ax\_9\_7\_1$).

      10. From the facts $C \neq C$ we get contradiction.

      11. Assume that: $C \neq D$.

      12. From the facts $one\_side(A, B, C, D)$ there exist a point F where $C \neq D$ and $two\_sides(A, F, C, D)$ and $two\_sides(B, F, C, D)$ (using $ax\_9\_7\_1$).

      13. From the facts $two\_sides(B, F, C, D)$ and $one\_side(B, E, C, D)$ it holds that $two\_sides(E, F, C, D)$ (using $th\_9\_8\_2$).

14. From the facts $two\_sides(A, F, C, D)$ and $two\_sides(E, F, C, D)$ it holds that $one\_side(A, E, C, D)$ (using $th\_9\_8\_1$).

15. From the facts $one\_side(A, E, C, D)$ we proved a conjecture.

QED

Theorem 145 (th_9_17.) *Assuming that $one\_side(C, D, A, B)$ and $bet(C, E, D)$ it holds that $one\_side(E, C, A, B)$.*

Theorem 146 (th_9_18_1.) *Assuming that $A \neq B$ and $point\_on\_line(C, A, B)$ and $col(D, E, C)$ and $two\_sides(D, E, A, B)$ it holds that $bet(D, C, E)$ and $not\_point\_on\_line(D, A, B)$ and $not\_point\_on\_line(E, A, B)$.*

Theorem 147 (th_9_18_2.) *Assuming that $A \neq B$ and $point\_on\_line(C, A, B)$ and $col(D, E, C)$ and $bet(D, C, E)$ and $not\_point\_on\_line(D, A, B)$ and $not\_point\_on\_line(E, A, B)$ it holds that $two\_sides(D, E, A, B)$.*

Proof:

1. From the facts $A \neq B$ and $not\_point\_on\_line(D, A, B)$ and $not\_point\_on\_line(E, A, B)$ and $point\_on\_line(C, A, B)$ and $bet(D, C, E)$ it holds that $two\_sides(D, E, A, B)$ (using $ax\_9\_1\_2$).

2. From the facts $two\_sides(D, E, A, B)$ we proved a conjecture.

QED

Theorem 148 (th_9_19_1.) *Assuming that $A \neq B$ and $point\_on\_line(C, A, B)$ and $col(D, E, C)$ and $one\_side(D, E, A, B)$ it holds that $out(C, D, E)$ and $not\_point\_on\_line(D, A, B)$.*

Theorem 149 (th_9_19_2.) *Assuming that $A \neq B$ and $point\_on\_line(C, A, B)$ and $col(D, E, C)$ and $out(C, D, E)$ and $not\_point\_on\_line(D, A, B)$ it holds that $one\_side(D, E, A, B)$.*

Proof:

1. From the facts $A \neq B$ and $not\_point\_on\_line(D, A, B)$ there exist a point F where $two\_sides(D, F, A, B)$ (using th_9_10).

2. From the facts $A \neq B$ and $two\_sides(D, F, A, B)$ and $point\_on\_line(C, A, B)$ and $out(C, D, E)$ it holds that $two\_sides(E, F, A, B)$ (using th_9_5).

3. From the facts $two\_sides(D, F, A, B)$ and $two\_sides(E, F, A, B)$ it holds that $one\_side(D, E, A, B)$ (using $th\_9\_8\_1$).

4. From the facts $one\_side(D, E, A, B)$ we proved a conjecture.

QED

Theorem 150 (th_9_31.) *Assuming that $one\_side(A, B, C, D)$ and $one\_side(A, D, C, B)$ it holds that $two\_sides(D, B, C, A)$.*

Chapter 10

Line reflexivity

Theorem 151 (th_10_2.) *Assuming that $A \neq B$ there exist point D, such that $is\_image(D, C, A, B)$.*

Theorem 152 (th_10_4.) *Assuming that $is\_image(B, A, C, D)$ it holds that $is\_image(A, B, C, D)$.*

Theorem 153 (th_10_5.) *Assuming that $A \neq B$ and $is\_image(D, C, A, B)$ and $is\_image(E, D, A, B)$ it holds that $C = E$.*

Theorem 154 (th_10_6_1.) *Assuming that $A \neq B$ there exist point D, such that $is\_image(C, D, A, B)$.*

Proof:
1. From the facts $A \neq B$ there exist a point F where $is\_image(F, C, A, B)$ (using th_10_2).
2. From the facts $is\_image(F, C, A, B)$ it holds that $is\_image(C, F, A, B)$ (using th_10_4).
3. From the facts $is\_image(C, F, A, B)$ we proved a conjecture.
QED

Theorem 155 (th_10_6_2.) *Assuming that $A \neq B$ and $is\_image(C, D, A, B)$ and $is\_image(C, E, A, B)$ it holds that $D = E$.*

Proof:
1. From the facts $is\_image(C, E, A, B)$ it holds that $is\_image(E, C, A, B)$ (using th_10_4).
2. From the facts $A \neq B$ and $is\_image(C, D, A, B)$ and $is\_image(E, C, A, B)$ it holds that $D = E$ (using th_10_5).
3. From the facts $D = E$ we proved a conjecture.

QED

Theorem 156 (th_10_7.) *Assuming that* $A \neq B$ *and* $is\_image(C, D, A, B)$ *and* $is\_image(C, E, A, B)$ *it holds that* $D = E$.

    Proof:

1. From the facts $A \neq B$ and $is\_image(C, D, A, B)$ and $is\_image(C, E, A, B)$ it holds that $D = E$ (using *th_10_6_2*).
2. From the facts $D = E$ we proved a conjecture.

QED

Theorem 157 (th_10_8_1.) *Assuming that* $A \neq B$ *and* $is\_image(C, C, A, B)$ *it holds that* $point\_on\_line(C, A, B)$.

Theorem 158 (th_10_8_2.) *Assuming that* $A \neq B$ *and* $point\_on\_line(C, A, B)$ *it holds that* $is\_image(C, C, A, B)$.

    Proof:

1. It holds that $is\_midpoint(C, C, C)$ (using *th_7_3_2*).
2. From the facts $A \neq B$ and $is\_midpoint(C, C, C)$ and $point\_on\_line(C, A, B)$ it holds that $is\_image\_spec(C, C, A, B)$ (using *ax_10_3_2_2*).
3. From the facts $A \neq B$ and $is\_image\_spec(C, C, A, B)$ it holds that $is\_image(C, C, A, B)$ (using *ax_10_3_4_1*).
4. From the facts $is\_image(C, C, A, B)$ we proved a conjecture.

QED

Theorem 159 (th_10_10.) *Assuming that* $A \neq B$ *and* $is\_image(E, C, A, B)$ *and* $is\_image(F, D, A, B)$ *it holds that* $CD \cong EF$.

Theorem 160 (th_10_12.) *Assuming that* $per(A, B, C)$ *and* $per(D, E, F)$ *and* $AB \cong DE$ *and* $BC \cong EF$ *it holds that* $AC \cong DF$.

Theorem 161 (th_10_14.) *Assuming that* $A \neq B$ *and* $is\_image(D, C, A, B)$ *and* $not\_point\_on\_line(C, A, B)$ *it holds that* $two\_sides(C, D, A, B)$.

Theorem 162 (th_10_15.) *Assuming that* $A \neq B$ *and* $point\_on\_line(C, A, B)$ *and* $not\_point\_on\_line(D, A, B)$ *there exist point* E, *such that* $perp(A, B, E, C)$ *and* $one\_side(E, D, A, B)$.

Theorem 163 (th_10_16_1.) *Assuming that* $not\_col(A, B, C)$ *and* $not\_col(D, E, F)$ *and* $AB \cong DE$ *there exist point* G, *such that* $cong3(A, B, C, D, E, G)$ *and* $one\_side(G, F, D, E)$.

Theorem 164 (th_10_16_2.) *Assuming that* $not\_col(A, B, C)$ *and* $not\_col(D, E, F)$ *and* $AB \cong DE$ *and* $cong3(A, B, C, D, E, G)$ *and* $one\_side(G, F, D, E)$ *and* $cong3(A, B, C, D, E, I)$ *and* $one\_side(I, F, D, E)$ *it holds that* $G = I$.

Chapter 11

Congruence and Comparison of Angles, Congruence theorem, Orthogonality for Subspaces

Theorem 165 (th_11_3_1.) *Assuming that cong_angle(A, B, C, D, E, F) there exist point G, point I, point J, point K, such that out(B, G, A) and out(B, I, C) and out(E, J, D) and out(E, K, F) and cong3(G, B, I, J, E, K).*

Theorem 166 (th_11_3_2.) *Assuming that out(B, G, A) and out(B, I, C) and out(E, J, D) and out(E, K, F) and cong3(G, B, I, J, E, K) it holds that cong_angle(A, B, C, D, E, F).*

Theorem 167 (th_11_4_1.) *Assuming that cong_angle(A, B, C, D, E, F) and $A \neq B$ and $C \neq B$ and $D \neq E$ and $F \neq E$ and out(B, G, A) and out(B, I, C) and out(E, J, D) and out(E, K, F) and $BG \cong EJ$ and $BI \cong EK$ it holds that $GI \cong JK$.*

Theorem 168 (th_11_4_2.) *Assuming that $A \neq B$ and $C \neq B$ and $D \neq E$ and $F \neq E$ and out(B, G, A) and out(B, I, C) and out(E, J, D) and out(E, K, F) and $BG \cong EJ$ and $BI \cong EK$ and $GI \cong JK$ it holds that cong_angle(A, B, C, D, E, F).*

Theorem 169 (th_11_6.) *Assuming that $A \neq B$ and $C \neq B$ it holds that cong_angle(A, B, C, A, B, C).*

Theorem 170 (th_11_7.) *Assuming that cong_angle(A, B, C, D, E, F) it holds that cong_angle(D, E, F, A, B, C).*

Theorem 171 (th_11_8.) *Assuming that cong_angle(A, B, C, D, E, F) and cong_angle(D, E, F, G, I, J) it holds that cong_angle(A, B, C, G, I, J).*

Theorem 172 (th_11_9.) *Assuming that $A \neq B$ and $C \neq B$ it holds that cong_angle(A, B, C, C, B, A).*

Theorem 173 (th_11_10.) *Assuming that cong_angle(A, B, C, D, E, F) and out(B, G, A) and out(B, I, C) and out(E, J, D) and out(E, K, F) it holds that cong_angle(G, B, I, J, E, K).*

Theorem 174 (th_11_13.) *Assuming that cong_angle(A, B, C, D, E, F) and bet(A, B, G) and $G \neq B$ and bet(D, E, I) and $I \neq E$ it holds that cong_angle(G, B, C, I, E, F).*

Theorem 175 (th_11_14.) *Assuming that bet(A, B, D) and distinct(A, B, D) and bet(C, B, E) and distinct(C, B, E) it holds that cong_angle(A, B, C, D, B, E).*

Theorem 176 (th_11_15_1.) *Assuming that not_col(A, B, C) and not_col(D, E, F) there exist point G, such that cong_angle(A, B, C, D, E, G) and one_side(G, F, E, D).*

Theorem 177 (th_11_15_2.) *Assuming that not_col(A, B, C) and not_col(D, E, F) and cong_angle(A, B, C, D, E, G) and one_side(G, F, E, D) and cong_angle(A, B, C, D, E, I) and one_side(I, F, E, D) it holds that $G = I$.*

Theorem 178 (th_11_16.) *Assuming that per(A, B, C) and $A \neq B$ and $C \neq B$ and per(D, E, F) and $D \neq E$ and $F \neq E$ it holds that cong_angle(A, B, C, D, E, F).*

Theorem 179 (th_11_17.) *Assuming that per(A, B, C) and cong_angle(A, B, C, D, E, F) it holds that per(D, E, F).*

Theorem 180 (th_11_18_1.) *Assuming that bet(A, B, C) and distinct(B, A, C) and $D \neq B$ and per(D, B, A) it holds that cong_angle(D, B, A, D, B, C).*

    Proof:

1. From the facts *distinct(B, A, C)* it holds that $B \neq A$ and $B \neq C$ and $A \neq C$ (using *ax_distinct_1*).
2. From the facts *per(D, B, A)* it holds that *per(A, B, D)* (using *th_8_2*).
3. From the facts *bet(A, B, C)* it holds that *bet(C, B, A)* (using *th_3_2*).
4. From the facts *bet(C, B, A)* it holds that *col(B, A, C)* (using *ax_4_10_4*).
5. From the facts $B \neq A$ it holds that $A \neq B$.

6. From the facts $per(A, B, D)$ and $A \neq B$ and $col(B, A, C)$ it holds that $per(C, B, D)$ (using th_8_3).

7. From the facts $per(C, B, D)$ it holds that $per(D, B, C)$ (using th_8_2).

8. From the facts $B \neq A$ it holds that $A \neq B$.

9. From the facts $B \neq C$ it holds that $C \neq B$.

10. From the facts $per(D, B, A)$ and $D \neq B$ and $A \neq B$ and $per(D, B, C)$ and $D \neq B$ and $C \neq B$ it holds that $cong\_angle(D, B, A, D, B, C)$ (using th_11_16).

11. From the facts $cong\_angle(D, B, A, D, B, C)$ we proved a conjecture.

QED

Theorem 181 (th_11_18_2.) *Assuming that $bet(C, B, D)$ and $distinct(B, C, D)$ and $A \neq B$ and $cong\_angle(A, B, C, A, B, D)$ it holds that $per(A, B, C)$.*

Theorem 182 (th_11_19.) *Assuming that $per(B, A, C)$ and $per(B, A, D)$ and $one\_side(C, D, A, B)$ it holds that $out(A, C, D)$.*

Theorem 183 (th_11_21_1_1.) *Assuming that $out(B, A, C)$ and $cong\_angle(A, B, C, D, E, F)$ it holds that $out(E, D, F)$.*

Theorem 184 (th_11_21_1_2.) *Assuming that $out(B, A, C)$ and $out(E, D, F)$ it holds that $cong\_angle(A, B, C, D, E, F)$.*

Theorem 185 (th_11_21_2_1.) *Assuming that $bet(A, B, C)$ and $distinct(A, B, C)$ and $cong\_angle(A, B, C, D, E, F)$ it holds that $bet(D, E, F)$ and $distinct(D, E, F)$.*

Theorem 186 (th_11_21_2_2.) *Assuming that $bet(A, B, C)$ and $distinct(A, B, C)$ and $bet(D, E, F)$ and $distinct(D, E, F)$ it holds that $cong\_angle(A, B, C, D, E, F)$.*

Theorem 187 (th_11_22_1.) *Assuming that $two\_sides(A, C, B, D)$ and $two\_sides(E, G, F, I)$ and $cong\_angle(A, B, D, E, F, I)$ and $cong\_angle(D, B, C, I, F, G)$ it holds that $cong\_angle(A, B, C, E, F, G)$.*

Theorem 188 (th_11_22_2.) *Assuming that $one\_side(A, C, B, D)$ and $one\_side(E, G, F, I)$ and $cong\_angle(A, B, D, E, F, I)$ and $cong\_angle(D, B, C, I, F, G)$ it holds that $cong\_angle(A, B, C, E, F, G)$.*

Theorem 189 (th_11_24.) *Assuming that $point\_in\_angle(A, B, C, D)$ it holds that $point\_in\_angle(A, D, C, B)$.*

Theorem 190 (th_11_25.) *Assuming that $point\_in\_angle(A, B, C, D)$ and $out(C, E, B)$ and $out(C, F, D)$ and $out(C, G, A)$ it holds that $point\_in\_angle(G, E, C, F)$.*

Theorem 191 (th_11_28.) *Assuming that $cong3(A, B, C, D, E, F)$ and $col(A, C, G)$ there exist point I, such that $cong4(A, B, C, G, D, E, F, I)$.*

Theorem 192 (th_11_29_1.) *Assuming that $le\_angle(A, B, C, D, E, F)$ there exist point G, such that $point\_in\_angle(C, A, B, G)$ and $cong\_angle(A, B, G, D, E, F)$.*

Theorem 193 (th_11_29_2.) *Assuming that $point\_in\_angle(C, A, B, G)$ and $cong\_angle(A, B, G, D, E, F)$ it holds that $le\_angle(A, B, C, D, E, F)$.*

Theorem 194 (th_11_30.) *Assuming that $le\_angle(A, B, C, D, E, F)$ and $cong\_angle(A, B, C, G, I, J)$ and $cong\_angle(D, E, F, K, L, M)$ it holds that $le\_angle(G, I, J, K, L, M)$.*

Theorem 195 (th_11_31_1.) *Assuming that $out(B, A, C)$ and $D \neq E$ and $F \neq E$ it holds that $le\_angle(A, B, C, D, E, F)$.*

Theorem 196 (th_11_31_2.) *Assuming that $A \neq B$ and $C \neq B$ and $bet(D, E, F)$ and $distinct(D, E, F)$ it holds that $le\_angle(A, B, C, D, E, F)$.*

Theorem 197 (th_11_32.) *Assuming that $A \neq B$ and $C \neq B$ it holds that $le\_angle(A, B, C, A, B, C)$.*

Proof:
1. It holds that $bet(A, C, C)$ (using th_3_1).
2. From the facts $C \neq B$ it holds that $out(B, C, C)$ (using th_6_5).
3. From the facts $A \neq B$ and $C \neq B$ it holds that $cong\_angle(A, B, C, A, B, C)$ (using th_11_6).
4. From the facts $A \neq B$ and $C \neq B$ and $C \neq B$ and $bet(A, C, C)$ and $out(B, C, C)$ it holds that $point\_in\_angle(C, A, B, C)$ (using $ax\_11\_23\_2$).
5. From the facts $point\_in\_angle(C, A, B, C)$ and $cong\_angle(A, B, C, A, B, C)$ it holds that $le\_angle(A, B, C, A, B, C)$ (using $ax\_11\_27\_2$).
6. From the facts $le\_angle(A, B, C, A, B, C)$ we proved a conjecture.
QED

Theorem 198 (th_11_33.) *Assuming that $le\_angle(A, B, C, D, E, F)$ and $le\_angle(D, E, F, G, I, J)$ it holds that $le\_angle(A, B, C, G, I, J)$.*

Theorem 199 (th_11_34.) *Assuming that $le\_angle(A, B, C, D, E, F)$ and $le\_angle(D, E, F, A, B, C)$ it holds that $cong\_angle(A, B, C, D, E, F)$.*

Theorem 200 (th_11_35.) *Assuming that $A \neq B$ and $C \neq B$ and $D \neq E$ and $F \neq E$ it holds that $le\_angle(A, B, C, D, E, F)$ or $le\_angle(D, E, F, A, B, C)$.*

Theorem 201 (th_11_36_1.) *Assuming that $bet(A, B, G)$ and $distinct(A, B, G)$ and $bet(D, E, I)$ and $distinct(D, E, I)$ and $le\_angle(A, B, C, D, E, F)$ it holds that $le\_angle(I, E, F, G, B, C)$.*

Theorem 202 (th_11_36_2.) *Assuming that $bet(A, B, G)$ and $distinct(A, B, G)$ and $bet(D, E, I)$ and $distinct(D, E, I)$ and $le\_angle(I, E, F, G, B, C)$ it holds that $le\_angle(A, B, C, D, E, F)$.*

Theorem 203 (th_11_41.) *Assuming that $not\_col(A, B, C)$ and $bet(B, A, D)$ and $D \neq A$ it holds that $lt\_angle(A, C, B, C, A, D)$ and $lt\_angle(A, B, C, C, A, D)$.*

Theorem 204 (th_11_43_1.) *Assuming that $not\_col(A, B, C)$ and $per(B, A, C)$ it holds that $acute(A, B, C)$ and $acute(A, C, B)$.*

Theorem 205 (th_11_43_2.) *Assuming that $not\_col(A, B, C)$ and $obtuse(B, A, C)$ it holds that $acute(A, B, C)$ and $acute(A, C, B)$.*

Theorem 206 (th_11_44_1.) *Assuming that $not\_col(A, B, C)$ and $AB \cong AC$ it holds that $cong\_angle(A, C, B, A, B, C)$.*

Proof:

1. It holds that $col(A, A, B)$ (using th_4_12).
2. It holds that $col(A, A, C)$ (using th_4_12).
3. It holds that $col(B, B, A)$ (using th_4_12).
4. From the facts $col(A, A, B)$ it holds that $col(A, B, A)$ and $col(B, A, A)$ and $col(B, A, A)$ and $col(A, A, B)$ and $col(A, B, A)$ (using th_4_11).
5. From the facts $col(A, A, C)$ it holds that $col(A, C, A)$ and $col(C, A, A)$ and $col(C, A, A)$ and $col(A, A, C)$ and $col(A, C, A)$ (using th_4_11).
6. From the facts $col(B, B, A)$ it holds that $col(B, A, B)$ and $col(A, B, B)$ and $col(A, B, B)$ and $col(B, B, A)$ and $col(B, A, B)$ (using th_4_11).
7. From the facts $AB \cong AC$ it holds that $AB \cong CA$ (using th_2_5).
8. From the facts $AB \cong CA$ it holds that $CA \cong AB$ (using th_2_2).
9. From the facts $CA \cong AB$ it holds that $CA \cong BA$ (using th_2_5).
10. It holds that $BC \cong CB$ (using ax_1).
11. From the facts $BC \cong CB$ it holds that $CB \cong BC$ (using th_2_2).
12. It holds that $A = B$ or $A \neq B$ (using ax_g1).
    13. Assume that: $A = B$.
        14. From the facts $col(A, A, C)$ and $A = B$ it holds that $col(A, B, C)$.
        15. From the facts $not\_col(A, B, C)$ and $col(A, B, C)$ we get contradiction (using ax_false_col).

16. Assume that: $A \neq B$.

  17. From the facts $A \neq B$ it holds that $out(B, A, A)$ (using th_6_5).

18. It holds that $A = C$ or $A \neq C$ (using ax_g1).

  19. Assume that: $A = C$.

    20. From the facts $col(A, B, A)$ and $A = C$ it holds that $col(A, B, C)$.

    21. From the facts $not\_col(A, B, C)$ and $col(A, B, C)$ we get contradiction (using ax_false_col).

  22. Assume that: $A \neq C$.

    23. From the facts $A \neq C$ it holds that $out(C, A, A)$ (using th_6_5).

    24. It holds that $B = C$ or $B \neq C$ (using ax_g1).

    25. Assume that: $B = C$.

      26. From the facts $col(A, B, B)$ and $B = C$ it holds that $col(A, B, C)$.

      27. From the facts $not\_col(A, B, C)$ and $col(A, B, C)$ we get contradiction (using ax_false_col).

    28. Assume that: $B \neq C$.

      29. From the facts $B \neq C$ it holds that $out(C, B, B)$ (using th_6_5).

      30. From the facts $B \neq C$ it holds that $C \neq B$.

      31. From the facts $C \neq B$ it holds that $out(B, C, C)$ (using th_6_5).

      32. From the facts $B \neq C$ it holds that $C \neq B$.

      33. From the facts $A \neq C$ and $B \neq C$ and $A \neq B$ and $C \neq B$ and $out(C, A, A)$ and $out(C, B, B)$ and $out(B, A, A)$ and $out(B, C, C)$ and $CA \cong BA$ and $CB \cong BC$ and $AB \cong AC$ it holds that $cong\_angle(A, C, B, A, B, C)$ (using $th\_11\_4\_2$).

      34. From the facts $cong\_angle(A, C, B, A, B, C)$ we proved a conjecture.

QED

Theorem 207 (th_11_44_2.) *Assuming that* $not\_col(A, B, C)$ *and* $cong\_angle(A, C, B, A, B, C)$ *it holds that* $AB \cong AC$.

Theorem 208 (th_11_44_3.) *Assuming that* $not\_col(A, B, C)$ *and* $lt(A, B, A, C)$ *it holds that* $lt\_angle(A, C, B, A, B, C)$.

Theorem 209 (th_11_44_4.) *Assuming that* $not\_col(A, B, C)$ *and* $lt\_angle(A, C, B, A, B, C)$ *it holds that* $lt(A, B, A, C)$.

Theorem 210 (th_11_46_1.) *Assuming that* $not\_col(A, B, C)$ *and* $per(B, A, C)$ *it holds that* $lt(A, B, B, C)$ *and* $lt(A, C, B, C)$.

Theorem 211 (th_11_46_2.) *Assuming that* $not\_col(A, B, C)$ *and* $obtuse(B, A, C)$ *it holds that* $lt(A, B, B, C)$ *and* $lt(A, C, B, C)$.

Theorem 212 (th_11_47.) *Assuming that $per(A, C, B)$ and $perp\_in(D, C, D, A, B)$ it holds that $bet(A, D, B)$ and $distinct(A, D, B)$.*

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Theorem 213 (th_11_49_1.) *Assuming that $cong\_angle(A, B, C, D, E, F)$ and $BA \cong ED$ and $BC \cong EF$ it holds that $AC \cong DF$.*

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Theorem 214 (th_11_49_2.) *Assuming that $cong\_angle(A, B, C, D, E, F)$ and $BA \cong ED$ and $BC \cong EF$ and $AC \cong DF$ and $A \neq C$ it holds that $cong\_angle(B, A, C, E, D, F)$ and $cong\_angle(B, C, A, E, F, D)$.*

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Theorem 215 (th_11_50_1.) *Assuming that $not\_col(A, B, C)$ and $cong\_angle(B, A, C, E, D, F)$ and $cong\_angle(A, B, C, D, E, F)$ and $AB \cong DE$ it holds that $AC \cong DF$ and $BC \cong EF$ and $cong\_angle(A, C, B, D, F, E)$.*

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Theorem 216 (th_11_50_2.) *Assuming that $not\_col(A, B, C)$ and $cong\_angle(B, C, A, E, F, D)$ and $cong\_angle(A, B, C, D, E, F)$ and $AB \cong DE$ it holds that $AC \cong DF$ and $BC \cong EF$ and $cong\_angle(B, A, C, E, D, F)$.*

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Theorem 217 (th_11_51.) *Assuming that $distinct(A, B, C)$ and $AB \cong DE$ and $AC \cong DF$ and $BC \cong EF$ it holds that $cong\_angle(B, A, C, E, D, F)$ and $cong\_angle(A, B, C, D, E, F)$ and $cong\_angle(B, C, A, E, F, D)$.*

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Theorem 218 (th_11_52.) *Assuming that $cong\_angle(A, B, C, D, E, F)$ and $AC \cong DF$ and $BC \cong EF$ and $le(B, C, A, C)$ it holds that $BA \cong ED$ and $cong\_angle(B, A, C, E, D, F)$ and $cong\_angle(B, C, A, E, F, D)$.*

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Theorem 219 (th_11_53.) *Assuming that $per(A, D, C)$ and $C \neq D$ and $distinct(A, B, D)$ and $bet(D, A, B)$ it holds that $lt\_angle(D, B, C, D, A, C)$ and $lt(A, C, B, C)$.*

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Theorem 220 (th_11_57.) *Assuming that $one\_side(B, E, A, D)$ and $one\_side(C, F, A, D)$ and $per(B, A, D)$ and $per(C, A, D)$ and $per(E, D, A)$ and $per(F, D, A)$ it holds that $cong\_angle(B, A, C, E, D, F)$.*

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Chapter 12

Parallelism (Euclidean Sense)

Theorem 221 (th_12_4.) *Assuming that $A \neq B$ it holds that $parallel\_broad(A, B, A, B)$.*

    Proof:

1. From the facts $A \neq B$ it holds that $point\_on\_line(A, A, B)$ and $point\_on\_line(B, A, B)$ and $same\_lines(A, B, B, A)$ (using th_6_17).

2. From the facts $A \neq B$ it holds that $B \neq A$.

3. From the facts $A \neq B$ and $B \neq A$ and $point\_on\_line(B, A, B)$ it holds that $same\_lines(A, B, A, B)$ (using th_6_16).

4. From the facts $A \neq B$ and $A \neq B$ and $same\_lines(A, B, A, B)$ it holds that $parallel\_broad(A, B, A, B)$ (using $ax\_12\_3\_3$).

5. From the facts $parallel\_broad(A, B, A, B)$ we proved a conjecture.

QED

Theorem 222 (th_12_5_1.) *Assuming that $parallel(A, B, C, D)$ it holds that $parallel(C, D, A, B)$.*

Theorem 223 (th_12_5_2.) *Assuming that $parallel\_broad(A, B, C, D)$ it holds that $parallel\_broad(C, D, A, B)$.*

Theorem 224 (th_12_6.) *Assuming that $parallel(A, B, C, D)$ and $point\_on\_line(E, C, D)$ and $point\_on\_line(F, C, D)$ it holds that $one\_side(E, F, A, B)$.*

Theorem 225 (th_12_9.) *Assuming that $perp(A, B, E, F)$ and $perp(C, D, E, F)$ it holds that $parallel\_broad(A, B, C, D)$.*

Theorem 226 (th_12_10.) *Assuming that $A \neq B$ there exist point D, point E, such that $D \neq E$ and $parallel\_broad(A, B, D, E)$ and $point\_on\_line(C, D, E)$.*

Theorem 227 (th_12_11.) *Assuming that $A \neq B$ and $not\_point\_on\_line(C, A, B)$ and $parallel\_broad(A, B, D, E)$ and $parallel\_broad(A, B, F, G)$ and $point\_on\_line(C, D, E)$ and $point\_on\_line(C, F, G)$ it holds that $same\_lines(D, E, F, G)$.*

Theorem 228 (th_12_15.) *Assuming that $parallel\_broad(A, B, C, D)$ and $parallel\_broad(C, D, E, F)$ it holds that $parallel\_broad(A, B, E, F)$.*

Theorem 229 (th_12_16.) *Assuming that $parallel\_broad(A, B, C, D)$ and $inter(G, E, F, A, B)$ there exist point I, such that $inter(I, E, F, C, D)$.*

Theorem 230 (th_12_17.) *Assuming that $is\_midpoint(E, A, C)$ and $is\_midpoint(E, B, D)$ and $A \neq B$ it holds that $parallel\_broad(A, B, C, D)$.*

Theorem 231 (th_12_18.) *Assuming that $AB \cong CD$ and $BC \cong DA$ and $not\_col(A, B, C)$ and $B \neq D$ and $col(A, E, C)$ and $col(B, E, D)$ it holds that $parallel\_broad(A, B, C, D)$ and $parallel\_broad(B, C, D, A)$ and $two\_sides(B, D, A, C)$ and $two\_sides(A, C, B, D)$.*

Theorem 232 (th_12_19.) *Assuming that $not\_col(A, B, C)$ and $parallel\_broad(A, B, C, D)$ and $parallel\_broad(B, C, D, A)$ it holds that $AB \cong CD$ and $BC \cong DA$ and $two\_sides(B, D, A, C)$ and $two\_sides(A, C, B, D)$.*

Theorem 233 (th_12_20.) *Assuming that $parallel\_broad(A, B, C, D)$ and $AB \cong CD$ and $two\_sides(B, D, A, C)$ it holds that $parallel\_broad(B, C, D, A)$ and $BC \cong DA$ and $two\_sides(A, C, B, D)$.*

Theorem 234 (th_12_21_1.) *Assuming that $two\_sides(B, D, A, C)$ and $parallel\_broad(A, B, C, D)$ it holds that $cong\_angle(B, A, C, D, C, A)$.*

Theorem 235 (th_12_21_2.) *Assuming that $two\_sides(B, D, A, C)$ and $cong\_angle(B, A, C, D, C, A)$ it holds that $parallel\_broad(A, B, C, D)$.*

Theorem 236 (th_12_22_1.) *Assuming that $out(E, A, C)$ and $one\_side(B, D, E, A)$ and $parallel\_broad(A, B, C, D)$ it holds that $cong\_angle(B, A, E, D, C, E)$.*

Theorem 237 (th_12_22_2.) *Assuming that $out(E, A, C)$ and $one\_side(B, D, E, A)$ and $cong\_angle(B, A, E, D, C, E)$ it holds that $parallel\_broad(A, B, C, D)$.*

Theorem 238 (th_12_23.) *Assuming that $not\_col(A, B, C)$ there exist point D, point E, such that $two\_sides(B, D, A, C)$ and $two\_sides(C, E, A, B)$ and $bet(D, A, E)$ and $cong\_angle(A, B, C, B, A, E)$ and $cong\_angle(A, C, B, C, A, D)$.*