Conjecture Synthesis for Inductive Theory Formation

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My Background

Automated inductive theorem proving in HOL. PhD at the University of Edinburgh (2009), now at Università degli Studi di Verona.

- **Case-Analysis for Rippling and Inductive Proof**.
  M. Johansson, L. Dixon and A. Bundy. Submitted to ITP 2010.
- **Lemma Discovery Techniques and Middle-Out Reasoning for Automated Inductive Proofs**.
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- **Conjecture Synthesis for Inductive Theories**.
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Introduction and Motivation

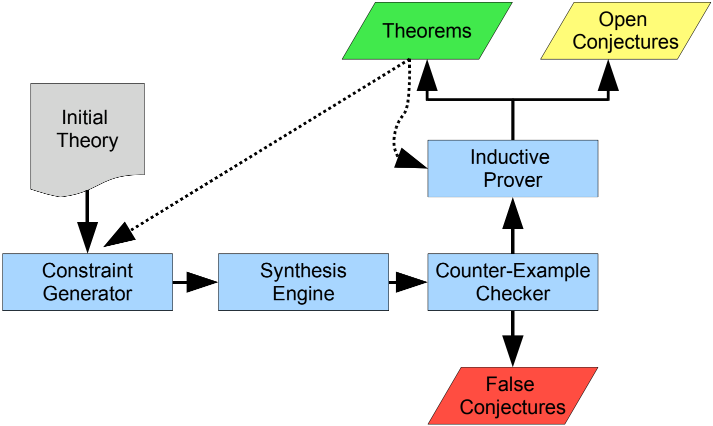
Induction: Reasoning about repetition, e.g. recursive datatypes and functions.
Challenge: Automate lemma discovery for (rewrite based) inductive proofs.

- Lemma typically need a separate inductive proof, not just an intermediate result.
- Generally assumed to require user intervention.
- Large libraries of previously proved theorems/lemmas e.g. Isabelle.
- Libraries insufficient for new theory developments.

IsaCoSy: Inductive Conjecture Synthesis

- Build conjectures from available functions, datatypes and variables.
- General: Can be applied to any recursive datatype defined in Isabelle without modification.
- Key idea for tractability: Turn rewriting upside-down.
  - Only generate irreducible terms.
- Enforced by constraints on term-synthesis. Avoid naive generate-and-test.
- Counter-example checking (Isabelle) + automatic inductive prover (IsaPlanner)
- New theorems provide more constraints.

Overview of IsaCoSy



Motivating Example: Definitions of List Reversal

Definition of *rev*:

$$rev([\,]) = [\,]$$

$$rev(h\#t) = rev(t)@[h]$$

Constraints on synthesis:

- Disallow [ ] to occur as argument of *rev*.
- Disallow # (cons) to occur as argument of *rev*.

Motivating Example: Distinctness for Lists

From definition of lists, Isabelle automatically derives:

$$[\,] \neq (h \# t)$$

Constraint on synthesis:

- Disallow $[\,]$ and $\#$ as simultaneous top-level arguments to opposite sides of an equality.

Motivating Example: Reflexivity

Reflexivity as a rewrite rule:

$$(x = x) = True$$

Constraint on synthesis:

- Disallow both sides of equality to be instantiated to the same term.

Motivating Example: Commutativity

Suppose we know that max is commutative:

$$max\ x\ y = max\ y\ x$$

Not a rewrite rule. Derive constraint on argument order:

- Measure of 1st argument $\geq$ measure of 2nd argument.
- Cuts out many symmetries.

Constraint Generation

- Initial constraints automatically derived from rules in input theory.
- Expressed in IsaCoSy's constraint language.
- Constraint from rule stored for its principal function symbol.

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Reflexivity: $(x = x) = True$

$UnEqual(arg_1, arg_2)$

List Distinctness: $[\,] \neq (h \# t)$

$NotAllowed(arg_1, [\,])$

$NotAllowed(arg_2, \#)$

Additional Heuristics

Can be configured by the user:

- Number of different variables. Default: $1 +$ max arity of functions.
- Where variables occur e.g. $Vars(RHS) \subseteq Vars(LHS)$
- Eagerly check for associativity and commutativity prior to synthesis.

The Synthesis Process

- Input: Initial constraints, max size of terms, user controlled heuristics.
- Start small: $\underbrace{?h_1}_{\textit{size } 1} = \underbrace{?h_2}_{\textit{size } 1}$
- Insert allowed constants and variables.
- After each size-iteration, counter-example check and prove.
- Generate new constraints from any new theorems.
- Increase term-size.

Evaluation

- Evaluated on theories about natural numbers, lists and binary trees.
- **Quality:** How does the set of theorems produced by IsaCoSy's compare to Isabelle's libraries?
- **Efficiency:** How much does IsaCoSy's heuristics improve over naive generate and test?

Natural Numbers

10/16 synthesised theorems are also in Isabelle's library:

$$a + 0 = a \qquad a + Suc\ b = Suc(a + b)$$
$$a * 0 = 0 \qquad a * Suc\ b = a + (a * b)$$
$$a + b = b + a \qquad a * b = b * a$$
$$(a + b) + c = a + (b + c) \qquad (a * b) * c = a * (b * c)$$
$$\left.\begin{array}{r}(a * b) + (c * b) = (a + c) * b \\ (a * b) + (a * c) = (b + c) * a\end{array}\right\} + \text{six variants not in library}$$

Isabelle's library contains another 2 theorems:

$$(Suc\ m) + n = m + (Suc\ n) \qquad x + (y + z) = y + (x + z)$$

Recall: 83%

Precision: 63%

Lists

9/24 synthesised theorems are also in Isabelle's library (with @ denoting append):

$$a \,@\, [\,] = a$$
$$rev(rev\ a) = a$$
$$rev(map\ a\ b) = map\ a(rev\ b)$$
$$(map\ a\ b) \,@\, (map\ a\ c) = map\ a\ (b \,@\, c)$$
$$foldl\ a\ (foldl\ a\ b\ c)\ d = foldl\ a\ b\ (c \,@\, d)$$
$$foldr\ a\ b\ (foldr\ a\ c\ d) = foldr\ a\ (b \,@\, c)\ d$$

$$(a \,@\, b) \,@\, c = a \,@\, (b \,@\, c)$$
$$(rev\ a) \,@\, (rev\ b) = rev\ (b \,@\, a)$$
$$len(rev\ a) = len\ a$$

$$\Big\} + 13 \text{ theorems}$$

- Isabelle's library contains only the 9 theorems above.
- Extra 13 theorems mostly about *rev* and *append*.

Recall: 100%

Precision: 38%

Binary Trees

Small theory about binary trees, involving functions *mirror*, *nodes* and *height*. No Isabelle library to compare.

$$mirror(mirror\ t) = t \qquad size(mirror\ t) = size\ t$$
$$height(mirror\ t) = height\ t \qquad max\ (size\ t)\ (height\ t) = size\ t$$

Run-times

- Compared to naive version: exponential cut in search space size.
- Synthesis generally takes a couple of hours, depending on maximum term size.
- Can cut run-times by restricting instantiation of type-variables for polymorphic datatypes (e.g. lists).
- Largest portion of time spent counter-example checking.

Future Directions and Applications

- Theory Library Formation:
  - Novel theory developments, generating routine library lemmas.
  - Generate benchmarks for inductive provers.
  - Generate libraries in Rich Model Language that can be shared between systems?
- Synthesis for generating/refining loop invariants.
  - *Refinement and Term Synthesis in Loop Invariant Generation. Maclean, Ireland, Atkey, Dixon. WING 2009*.

Conclusions & Summary

- IsaCoSy: Inductive theory formation by synthesis.
- Only generates irreducible terms, which keeps search space tractable.
- High recall, many interesting theorems synthesised.
- Lower precision. Too many variants of theorems generated.
- Using synthesised background theory increase power of prover. Mange to prove harder theorems automatically.
- `dream.inf.ed.ac.uk/projects/lemmadiscovery/`

Current Work in Verona

With Maria Paola Bonacina. Just starting:

- Extending SMT solver with F.O. reasoning: DPLL($\Gamma + T$).
  - DPLL is good at large conjunctions.
  - Rewrite based F.O. proves can handle quantifiers better.
- Contrast efficiency/expressiveness of logics. Typed/untyped settings.
- Extending combination of theories for F.O. provers.
- Combining non-stably infinite theories.