Deciding Non-linear Numerical Constraints: an Overview

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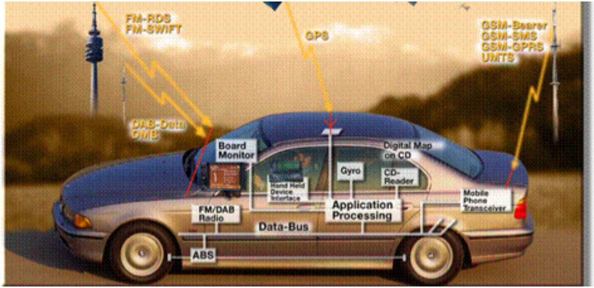
Motivation I

By far most micro-processors nowadays do not occur in desktop PC's but embedded in technical systems (trains, cars, robots, your washing machine etc.)



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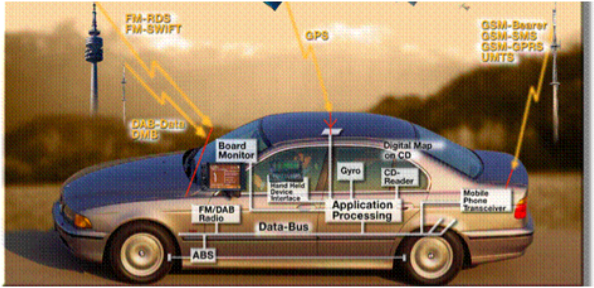

Models of technical systems usually in numerical domains.

Motivation II

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So: to solve discrete problem,
exploit corresponding continuous problem ("relaxation").

Example: MILP

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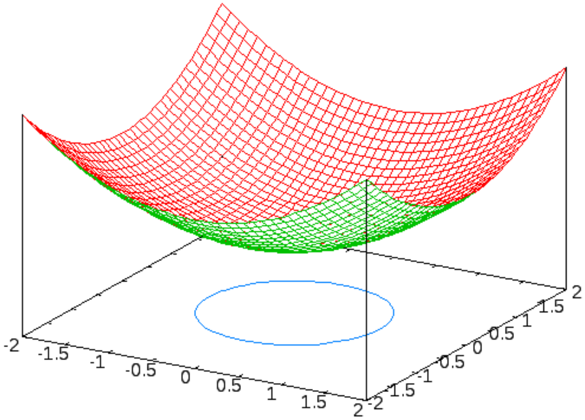

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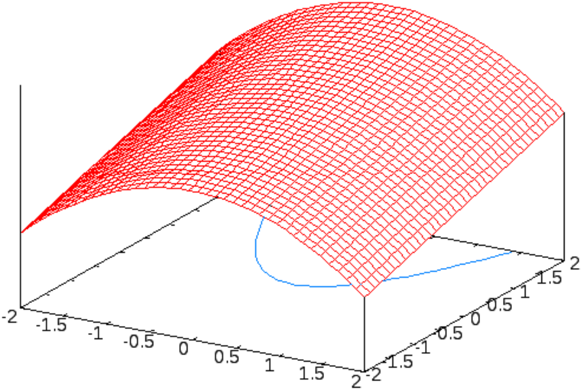

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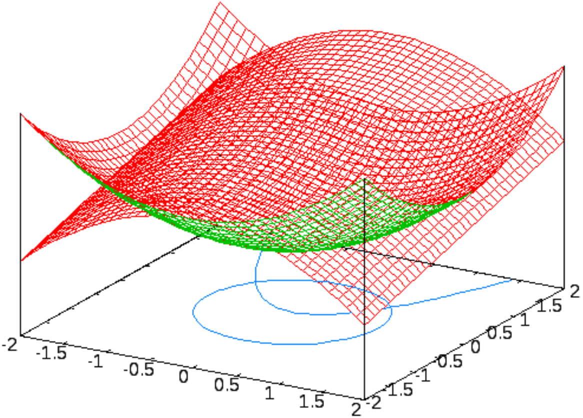

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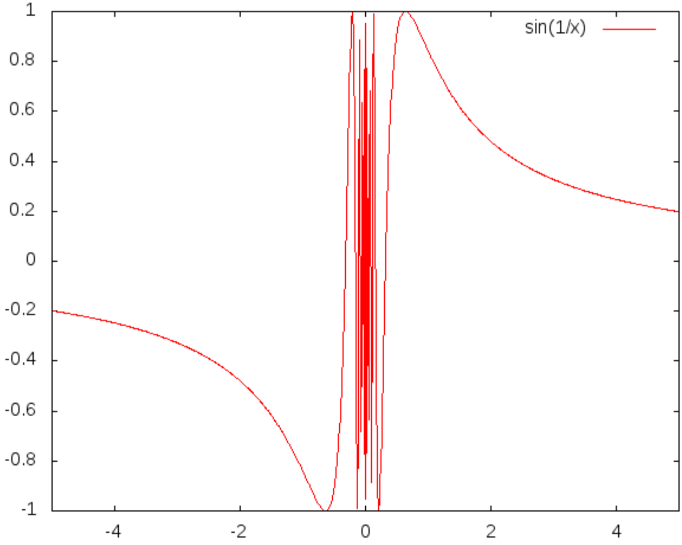

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Given: formula in certain sub-class of $FO(\mathbb{R}, =, \leq, <, +, \times, \text{sin}, \dots)$

Decide: `sat/unsat`

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Subclasses: quantifier-free, polynomial, linear, ...

Contents

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However: all univariate polynomials, and all polynomials with degree up to 2 can be written as SOS

Special Case: System of Polynomial equations

$f_1(x) = 0, \ldots, f_r(x) = 0$ does not have a solution iff there exist

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System of polynomial equations and inequalities:
Positivstellensatz [Stengle, 1974]

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What about certificates after adding sin, . . . ?

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Situation hopeless?

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Well known in numerical analysis (well-posed problems), but in the context of decidability questions new (independently introduced by several people since ~ 2000, usually called *robust problem*).

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Constraint ϕ *robust* iff
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Problem *quasi-decidable* iff
there is an algorithm that
correctly checks satisfiability and
terminates for all robust problem instances.

Quasi-decidability of $\mathbb{R}$

Theorem (Ratschan [2002, 2006])

$FO(\mathbb{R}, =, <, +, \times, \exp, \sin, \ldots)$ *is quasi-decidable.*

Assumptions:

- all variables bounded
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Implementation: `http://rsolver.sourceforge.net`

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non-satisfiability: statement over uncountable set,
symbolic representation needed

Branch and Bound

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Algorithm $BB(\phi, B)$: either returns $\texttt{unsat}$ or runs forever

$S \leftarrow test(\phi, B)$
if $S = \texttt{unsat}$ **then** S
else
  let B be such that $B = B_1 \cup B_2$,
  non-overlapping
  if $BB(\phi, B_1) = BB(\phi, B_2) = \texttt{unsat}$ **then** $\texttt{unsat}$

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Can be interleaved with a satisfiability test.

unsat test

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Interval arithmetic computes interval $f(I_1, \ldots, I_n)$ such that $\{f(x_1, \ldots, x_n) \mid x_1 \in I_1, \ldots, x_n \in I_n\} \subseteq f(I_1, \ldots, I_n)$

`unsat` test

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More powerful techniques based on

- advanced interval techniques [Neumaier, 1990, Moore et al., 2009],
- constraint propagation [Cleary, 1987, Jaulin et al., 2001],
- LP-relaxations [McCormick, 1976, Neumaier, 2004]

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Traditionally, computer science does not take into account perturbation, and assumes decision procedures.

Use quasi-decision procedures, that is, algorithms that need not terminate for non-robust inputs.

Literature I

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