Formalization of Wu's Simple Method in Coq

Jean-David Génevaux   Julien Narboux   Pascal Schreck

University of Strasbourg - LSIIT - CNRS

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Automatic Deduction in Geometry

Algebraic methods

- Gröbner bases [Kap86]
- Wu's method [Wu78, Cho85, Cho88, Wan01, Wan04]
- Geometric Algebra [LW00]

Synthetic

- Gelernter [Gel59]
- Deductive database [cCsGzZ00]
- The area method [CGZ94]
- Full angle method [CGZ96]

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Outline

An algebraic method

The initial goal is to show that:

$$\forall AB\ldots, h_1 \wedge \ldots \wedge h_k \Rightarrow g$$

This goal is translated into:

$$\forall \overrightarrow{x}, \qquad \bigwedge_i (h_i(\overrightarrow{x}) = 0) \Rightarrow (g(\overrightarrow{x}) = 0)$$

where h_i and g are multivariate polynomials in $\mathbb{F}(x_1, \ldots x_m)$.
We need to show that:

$$\bigcap_i Zeros(h_i) \subseteq Zero(g)$$

Noted:

$$\boxed{Zero(h_i) \subseteq Zero(g)}$$

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Only equalities $\rightarrow$ unordered geometry.

Which zeros ?

- For elementary geometry, we are interested in ensuring the set of real zeros of the hypothesis polynomials is contained in the set of real zeros of the conclusion polynomial.
- In practice, it often suffices to consider complex zeros instead of the real zeros, but not always.

(First) Incompleteness

Wu's method is incomplete as it considers only complex zeros.
For instance, $\forall x, y. x^2 + y^2 = 0 \implies x = 0 \wedge y = 0$ is true in $\mathbb{R}$ but not in $\mathbb{C}$.

Nullstellensatz

- If $\exists r,\; q_1, \ldots, q_k \;\; g^r = \sum_i q_i h_i$ then $Zero(h_i) \subseteq Zero(g)$.

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- Hilbert's *Nullstellensatz* theorem states that if $\mathbb{F}$ is algebraically closed, then the converse is also true:

$$\exists r,\ q_1, \ldots, q_k\ \ g^r = \sum_i q_i h_i \ \Leftrightarrow\ Zero(h_i) \subseteq Zero(g)$$

That is, we can *always* find such polynomials.

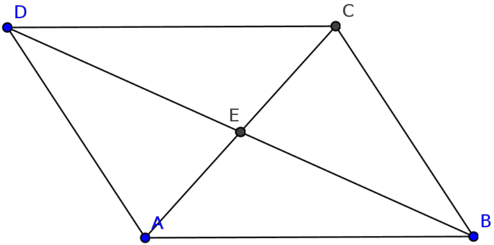
Example

Parallelogram

If $AB \parallel DC$ and $AD \parallel BC$ and $Col\ EAC$ and $Col\ EBD$

then

$$AE \equiv EC$$



Example

Parallelogram

If $AB \parallel DC$ and $AD \parallel BC$ and $Col\ EAC$ and $Col\ EBD$ and $\neg ColABC$

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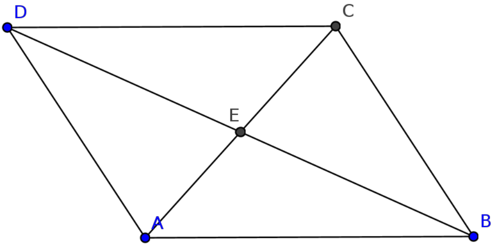


Non degeneracy conditions

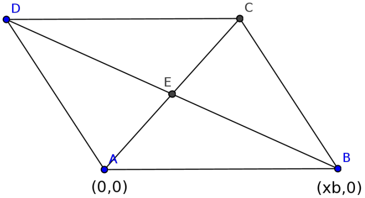
- Non degeneracy conditions (ndgs): $p(x) \neq 0$:
    - $A \neq B$
    - $\neg Col\ ABC$
    - $\neg Parallel\ ABCD$
    - ...
- Non degeneracy conditions is a central issue in formal geometry (see [DDS00, Nar08] for instance)
    - hard to find
    - proofs of degenerated cases are often difficult
- Wu's method generates non degeneracy conditions

$$\boxed{Zero(h_i) - \bigcup Zero(ndgs) \ \subseteq \ Zero(g)}$$

The main ideas:

1. Algebraization
   In practice the choice of a coordinate system is crucial.

$$
\begin{aligned}
h_1 &= -x_B * (y_D - y_C) \\
h_2 &= x_D * y_C + y_D * (x_B - x_C) \\
h_3 &= x_E * -y_C + y_E * x_C \\
h_4 &= (x_E - x_B) * y_D + \\
&\quad y_E * (x_B - x_D) \\
g &= x_E^2 + y_E^2 - \\
&\quad (x_E - x_C)^2 + (y_E - y_C)^2
\end{aligned}
$$



The main ideas:

1. Algebraization
   In practice the choice of a coordinate system is crucial.
2. Triangulation

   In general triangulation is slow, but constructive geometry statements are almost in triangular form.

x_A	y_A	x_B	y_B	x_C	y_C	x_D	y_D	x_E	y_E
x	x	x	x	x	x	x	x		
x	x	x	x	x	x	x	x		
x	x	x	x	x	x	x	x	x	x
x	x	x	x	x	x	x	x	x	x

The main ideas:

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   In practice the choice of a coordinate system is crucial.
2. Triangulation

   In general triangulation is slow, but constructive geometry statements are almost in triangular form.
3. Successive pseudo-division: divide the goal by the hypotheses

The main ideas:

1. Algebraization
   In practice the choice of a coordinate system is crucial.
2. Triangulation
   (using pseudo-division)
   In general triangulation is slow, but constructive geometry statements are almost in triangular form.
3. Successive pseudo-division: divide the goal by the hypotheses

The main character: pseudo-division

Pseudo-division of g by h in the variable v

$$a^k g = qh + r$$

where a is the leading coefficient of h and $deg(r, v) < deg(h, v)$.

Remark 1

If we know that $r = 0$ then :

$$\forall \overrightarrow{x}\ h(\overrightarrow{x}) = 0 \wedge a(\overrightarrow{x}) \neq 0 \implies g(\overrightarrow{x}) = 0$$

Remark 2

$r = prem(g, h)$ belongs to the ideal generated by g and h:

$$r = a^k \times g + (-q) \times h$$

Successive pseudo-division

$$sprem(g, [h_1, \ldots, h_n]) = sprem((prem_g(g, h_n), [h_1, \ldots, h_{n-1}])$$

$$a_1^{k_1} a_n^{k_n} g = q_1 h_1 + q_2 h_2 + \ldots + q_n h_n + r$$

Remark 1'

If we know that $r = 0$ then :

$$\forall \overrightarrow{x} \begin{array}{c} h_1(\overrightarrow{x}) = 0 \wedge \ldots \wedge h_n(\overrightarrow{x}) = 0 \\ \wedge \\ a_1(\overrightarrow{x}) \neq 0 \wedge \ldots \wedge a_n(\overrightarrow{x}) \neq 0 \end{array} \Longrightarrow g(\overrightarrow{x}) = 0$$

Completeness of the method

What does that mean if $r \neq 0$?

For some triangulation process (Ritt/Wu's characteristic sets) a theorem of Wu states that:

- Either the system is not irreducible
- or the theorem is generally false in metric geometry.

Available approaches

- Ltac
  - ☺ High-level language
  - ☹ Hard to debug
  - ☹ Generate large proofs terms
- Ocaml
  - ☺ Full programming language
  - ☺ Easier to debug
  - ☹ Need to generate proof term
- Coq itself (a reflexive approach)
  - ☺ Generate small proofs terms
  - Termination should be proved
- Using a certificate/validator approach
  - ☹ It is not possible to prove completeness
  - ☺ Certificate generator can be written using a different programming language, use heuristics, ...
  - ☺ Same validator can be used for several provers

Our choices:

- Ltac : algebraization, choice of a reference, simplification
- Triangulation and successive pseudo-division:
  - Ocaml: certificate generation
  - Coq: validator using reflection

Limitation

- We can only extract a (self certifying) prover for the *core* of Wu's method

Link between algebraic definition and synthetic geometry

In his Phd (12/2011), Tuan Minh Pham proves that:

```
Lemma transcol :
forall (A B C : Point), col A B C ->
(X A-X B)*(Y C-Y B)-(Y A-Y B)*(X C-X B)=0.
Lemma transparallel :
forall (A B C D : Point),
parallelLine (line A B ) (line C D) ->
(X B-X A)*(Y D-Y C)=(Y B-Y A)*(X D-X C).
Lemma transliesOn :
forall (A B C : Point),
liesOnLine A (line B C) ->
(X B-X A)*(Y C-Y B)-(Y B-Y A)*(X C-X B)=0.
```

```
Lemma transperpendicular :
forall (A B C D : Point),
 perpencicular (line A B) (line C D) ->
(X B-X A)*(X D-X C)+(Y B-Y A)*(Y D-Y C)=0 .
Lemma transsamedistance :=
 forall (A B C D : Point),
 distance A B = distance C D ->
(X B-X A)2 + (Y B-Y A)2=(X D-X C)2 + (Y D-Y C)2.
```

Choice of a coordinate system: the lemmas

In practice the choice of a convenient coordinate system is crucial. Following John Harrison's "Without loss of generality" [Har09], we show that the predicates are invariant under translation and rotation.

Example: collinear

```
Lemma collinear_inv_translation: forall A B C V,
 collinear A B C <->
 collinear (trans A V) (trans B V) (trans C V).

Lemma collinear_inv_rotation: forall A B C cos sin,
 cos*cos + sin*sin = 1 ->
 (collinear A B C <->
  collinear (rot A cos sin) (rot B cos sin) (rot C cos sin)).
```

Proofs can be done using ring/Gröbner basis.

Choice of a coordinate system: the tactic

The tactic

Algebraization O I H

The following predicates/functions are available:

collinear, parallel, orthogonal, midpoint, intersection of lines, square of length, equality of points, angles or lengths.

Limitations

- The tactic can not deal with user defined predicates automatically. Adding a new predicate requires to add the lemmas for invariance under translation and rotation and to update the tactic.

Design of the certificate

Main idea:

Provide r, l and $q_1, \ldots, q_k$ such that:

$$l \times g^r = \sum_i q_i \times h_i$$

Grégoire, Pottier, Théry's idea:

Use `let ... in` to compress this certificate using sharing (straight line programs).

$$\text{let } p_1 = q_1 * h_1 + q_2 * h_2 \text{ in}$$
$$\text{let } p_2 = q_3 * p_1 + q_4 * h_2 \text{ in}$$
$$\ldots$$

Generation of the certificate

We need a prover based on Wu's method which generate a *certificate*.

- Existing implementations either not open source or relying on proprietary CAS (Maple)[1].
- We aim at integration into Coq.

Hence we developed our own implementation of Wu's method in Ocaml based on a slightly optimized version of Loïc Pottier library for multivariate polynomials.

Second incompleteness

Our implementation is incomplete because we do not check polynomials for irreducibility (this requires factorization).

[1] OpenGeoProver was not available when we started this work.

Certificates generation

We generate certificates for:

1. The pseudo-division
2. The successive pseudo-division
3. Triangulation

Certificate generation I

1. We just need to keep the quotient and the lead coefficient:

$$r = a^k \times g + (-q) \times h$$

```
let pseudo_div_num g h x certif =
    let (r,a,k,q) = pseudo_div (g.p) (h.p) x in
    let new_n = new_num () in
      certif := (new_n, r, [(a^^k, g.n);(p_zero -- q, h.n)])
                ::(!certif);
      {p=r ; n= new_n}
```

Certificate generation II

2. We know that :

$$l \times g = \sum_i (t_i \times s_i) + r_{final}$$

where

$$s_i = q_i \times \prod_{j=1}^{i-1} c_j^{d_j}$$

3. The triangulation phase is based on pseudo-division.
Invariant: the polynomials are in the ideal generated by the hypotheses.

$$[h_1, \ldots, h_i, \ldots, h_j, \ldots, h_n] \rightarrow [h_1, \ldots, h_i, \ldots, prem(h_i, h_j, v), \ldots, h_n]$$

Certificate generation III

Remark

- To prove correctness we *do not need* to prove that the triangulation phase really triangulates.
- This shows that proving the method in Coq itself would not be so difficult.

Checking the certificate

Algorithm

- Computing $l \times g^r$ and $\sum_i q_i \times h_i$
- Checking equality using ring tactic normalization function

Reuse:

We reuse the validator proven correct by Grégoire, Pottier and Théry.

Technical limitation:

All shared polynomials must be in the ideal.

- Well-adapted to Gröbner basis
- Using Wu's method some other polynomials could be shared.

Benchmark I

Theorem	Wu / Caml	Wu / Coq	GB / Coq	Wu / GB
Pascal_2	0.013	21	-	-
Pascal_1	0.024	22	1652	$\times$75
Ptolemy95	0.010	10	30	$\times$3
Pappus	0.043	3	8	$\times$2.6
Altitudes	0.002	3	7	$\times$2.3
Simson	0.002	5	8	$\times$1.6
Perp-bisect	0.001	2	3	$\times$1.5
Pythagore	0.001	1	1	$\times$1
Feuerbach	0.038	15	15	$\times$1
Isoceles	0.001	1	1	$\times$1
Euler Line	0.063	9	6	$\times$0.6
Medians	0.001	3	2	$\times$0.6
Chords	0.015	4	2	$\times$0.5
Thales	0.003	6	3	$\times$0.5
Bissectors	0.001	6	3	$\times$0.5
Desargues	0.027	99	10	$\times$0.1
Ceva	0.025	98	6	$\times$0.06

Intel(R) Core(TM) i5 CPU 750 @ 2.67GHz with 4Gb RAM.

Benchmark II

Checking certificate using Ocaml:

- Between 1% and 80%
- On average: 50%

Conclusion

- Certificate based approaches are flexible: we could reuse Pottier et al. checker.
- But certificate checking time is significant.
- Wu's method and Gröbner basis seems to be complementary.

Perspectives

- Implement the full method of Wu.
- Add automatic geometrization.
- Add automatic choice of a reference.
- Other data-structure for certificates (pseudo-division ?).

Questions ?

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