Geometric constructions, first order logic and implementation

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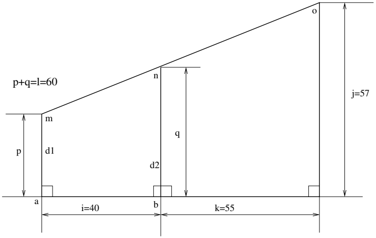
Some domains where geometric constructions (could) appear

- Education: Statement $\rightarrow$ program of construction

  Let d_1 and d_2 be 2 parallel lines, $A \in d_1$ and $B \in d_2$ be two points, and O be any point, how to construct a line Δ passing through O and meeting d_1 in M and d_2 in N such as $AM + BN = k$, (k is a given constant).

- Technical drawing: sketch $\rightarrow$ precise drawing



- Architecture, photogrammetry (projections $\rightarrow$ 3D-objects), curves et surfaces, molecule problem, robotic . . .

This talk is focused on the first domain.

Back to school

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Exercice

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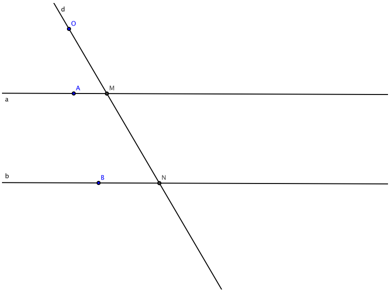

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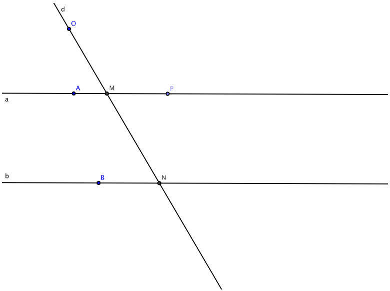

Let P be on d_1 at distance k from A

AM+MP= k =AM+BN

it is easy to see that (M, P, N, B) is a parallelogram

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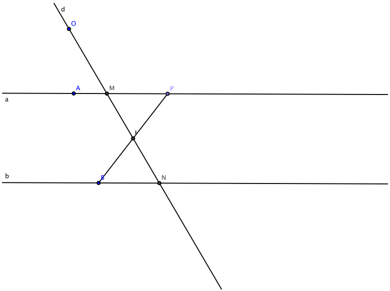

construction :

Draw point P on d1 at distance k from A

Construct point I as the midpoint of P and A

Draw Δ as line (OI)

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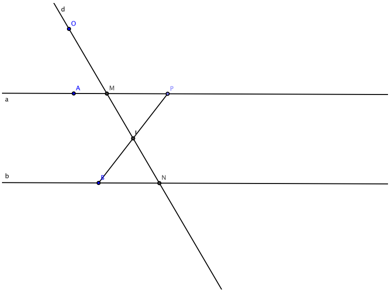
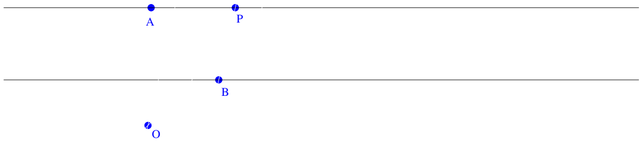


A, B, O, d_1, k : free
(A is on d_1)
$d_2 = \text{lpd}(B, \text{dir}(d_1))$
$P = \text{interlc}(d_1, \text{cir}(A,k))$
$I = \text{mid}(P,B)$
$\Delta = \text{lpp}(O,I)$

Testing the construction ...



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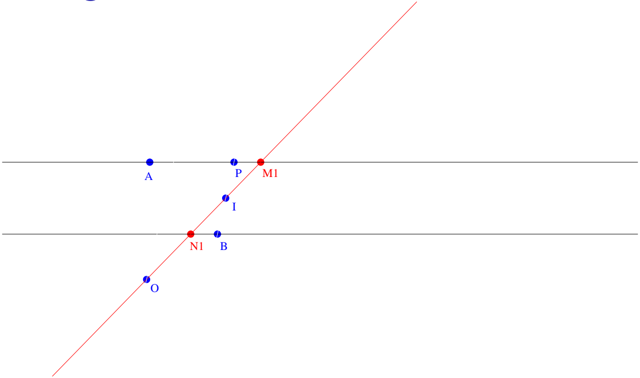

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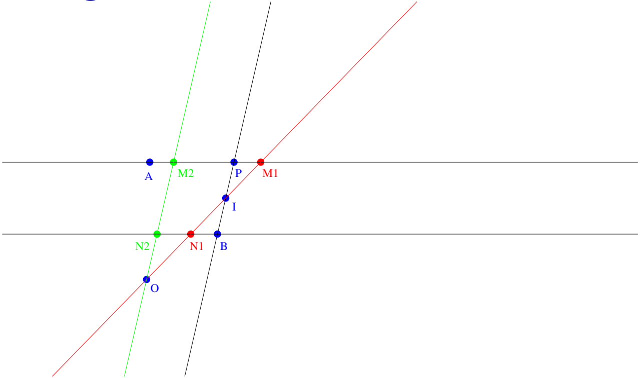

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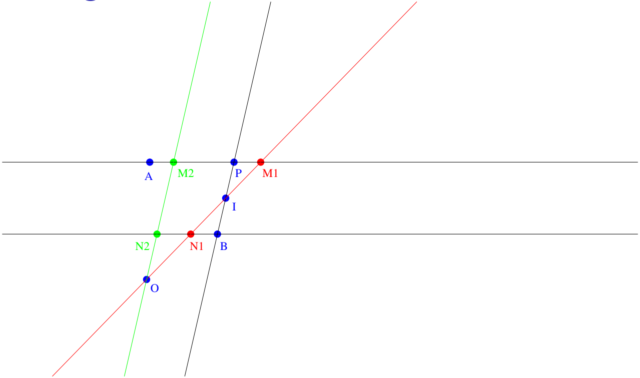


Explanation

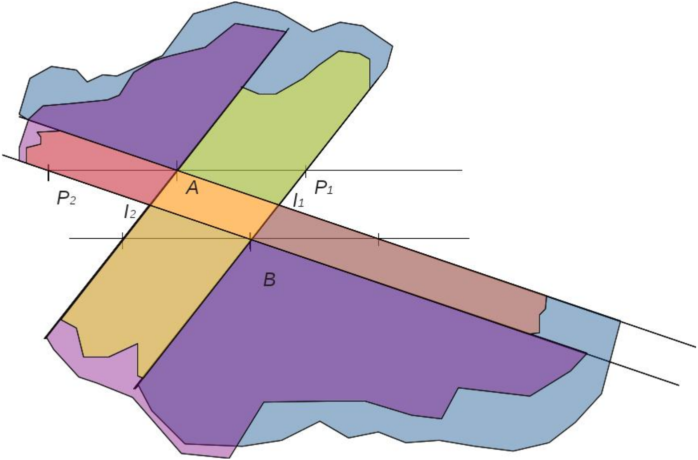
Point O being in this position, (M, P, N, B) is no more a parallelogram, but (M, P, B, N) is.
This leads to another construction where:
$\Delta = \text{lpd}(O,\text{dir}(\text{lpp}(P,B)))$.

Discussion (1)

We have two cases to consider, but there are other flaws :

$P = \text{interlc}(d_1, \text{cir}(A,k))$	there is two such points
$I = \text{mid}(P,B)$	ok
$\Delta = \text{lpp}(O,I)$	not defined if $O{=}I$

Discussion (2) ... a lot of cases



A *Program* of Construction

```
A, B, O, k, di : free
d1 = lpd(A, di)
d2 = lpd(B, di)
C = cir(A, k)
for P in interlc(d1, C)
  for case
    case pll(M,P,N,B):
     I = mid(P,B)
     if I <>O then
       Delta = lpp(O,I)
     else
         fail
     endif
```

```
case pll(M,P,B,N):
   if P <> B then
     d3 = lpp(P,B)
      di3 = dir(d3)
      Delta = lpd(O, di3)
   else
      fail
endif endcase endfor
```

Formalization and first order logic

Ruler and compass constructions

Definition

A point P is said RC-constructible from base points $\{B_0, \dots, B_k\}$ if there is a *finite* sequence of points $\{P_0, \dots, P_n\}$ such that each point P_i is either a base point, or a the intersection of lines or circles built from $\{P_0, \dots, P_{i-1}\}$ and $P = P_n$

Result

The problem of ruler and compass construction is not expressible in first order logic because of the notion of finiteness.

RC-construction and Tarski's elementary geometry

Quoting Tarski

For instance, the statement that every angle can be divided into three congruent angles is an elementary sentence in our sense [...]. On the other hand, the general notion of constructibility by rule and compass cannot be defined in elementary geometry, and therefore the statement that an angle in general cannot be trisected by rule and compass is not an elementary sentence.

Formalization of geometry

- Euclide, Hilbert
- Tarski
  Fact: Tarski's elementary geometry does not include RC constructions
- RC-constructible geometry (J. Duprat, Coq)
- Algebraic: the association of Wu (or Grobner basis) and Lebesgue's methods results into a decidability procedure (G. Chen implemented it in Maple)

We consider here an *ad hoc* formalization (in the same spirit than F. Guilhot did) in multi-sorted first order logic.

Syntactic considerations

An example of geometric signature

We have to consider something like that:

signature SIMP-SIGN-GEOM

sorts

`length`

`point`

`line`

`circle`

functional symbols

`dist`: `point point` $\rightarrow$ `length`

`radius`: `circle` $\rightarrow$ `length`

`interll`: `line line` $\rightarrow$ `point`

`intercl`: `circle line` $\rightarrow$ `point`

...

predicative symbols

`is-onl`: `point line` $\rightarrow$

`is-onc`: `point circle` $\rightarrow$

...

Signature and expressiveness

But ...

Problems

- partial functions
- multi-functions
- cases of figure

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A possible answer

- pre-conditions
- numbered functions
- *axioms with disjunctions*

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short discussion

pre-conditions + numbered functions vs relations ?

Signature and expressiveness

But ...

Problems

- partial functions
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- cases of figure

A possible answer

- pre-conditions
- numbered functions
- *axioms with disjunctions*

Expressiveness

```
mid(A,B) =
if A=B then A
else
interll(line(A,B),
line(intercc1(ccr(A,dist(A,B)), ccr(B, dist(A,B))),
intercc2(ccr(A,dist(A,B)), ccr(B,dist(A,B)))))
```

Constructibility vs construction

Constructibility

For a given constraint system $\mathcal{C}(\mathcal{X}, \mathcal{A})$, with unknowns $\mathcal{X}$and parameters $\mathcal{A}$, prove

$$\forall \mathcal{A} \exists \mathcal{X}, \mathcal{C}(\mathcal{X}, \mathcal{A})$$

Construction

For a given constraint system $\mathcal{C}(\mathcal{X}, \mathcal{A})$, with unknowns $\mathcal{X}$and parameters $\mathcal{A}$, find F such that,

$$\forall \mathcal{A}, \forall \mathcal{X}, \mathcal{C}(\mathcal{X}, \mathcal{A}) \Leftrightarrow \mathcal{X} = F(\mathcal{A})$$

Again, the geometric construction problem is out of the first order logic.

Logical expression of construction

In fact, the previous examples let you suspect, that it is a bit more complicated, we have to consider the bigger formula:

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$$
\begin{array}{l}
\forall \mathcal{A} \forall \mathcal{X} . \\
\left[\begin{array}{l}
\mathcal{C}(\mathcal{X}, \mathcal{A}) \\
\Leftrightarrow \\
\left(\begin{array}{l}
\quad (\delta_1(\mathcal{A}) \supset \mathcal{X} = F_{1,1}(\mathcal{A}) \vee \ldots \vee \mathcal{X} = F_{1,k_1}(\mathcal{A})) \\
\wedge (\delta_2(\mathcal{A}) \supset \mathcal{X} = F_{2,1}(\mathcal{A}) \vee \ldots \vee \mathcal{X} = F_{2,k_2}(\mathcal{A})) \\
\ldots \\
\wedge (\delta_l(\mathcal{A}) \supset \mathcal{X} = F_{l,1}(\mathcal{A}) \vee \ldots \vee \mathcal{X} = F_{l,k_l}(\mathcal{A})) \\
\wedge \ (\Delta(\mathcal{A}) \supset \ \Psi(\mathcal{X}, \mathcal{A})) \\
\wedge \ (\Omega(\mathcal{A}) \supset \bot)
\end{array} \right)
\end{array} \right] \\
\wedge (\delta_1(\mathcal{A}) \vee \ldots \vee \delta_l(\mathcal{A}) \vee \Delta(\mathcal{A}) \vee \Omega(\mathcal{A}))
\end{array}
$$

where all the predicative and functional terms but $\mathcal{C}$, are to be discovered.

Example (1)

$$
\begin{array}{l}
\forall c_1 : \texttt{circle},\ c_2 : \texttt{circle},\ x : \texttt{point}. \\
\left[\begin{array}{l}
(x\, \texttt{is-onc}\, c_1 \wedge x\, \texttt{is-onc}\, c_2) \\
\Leftrightarrow \\
\left(\begin{array}{l}
(\delta_1(c_1, c_2) \ \supset x = \texttt{intercc1}(c_1, c_2) \\
\qquad\qquad\qquad \vee\, x = \texttt{intercc2}(c_1, c_2)) \\
\wedge(c_1 = c_2 \ \supset\ x\ \texttt{is-onc}\ c_1) \\
\wedge(\neg\delta_1(c_1, c_2)\ \wedge c_1 \neq c_2 \supset \bot)
\end{array}\right)
\end{array}\right] \\
\wedge(\delta_1(c_1, c_2) \vee (\neg\delta_1(c_1, c_2) \wedge (c_1 \neq c_2)) \vee c_1 = c_2)
\end{array}
$$

where δ_1 is defined by:

$$\delta_1(c_1, c_2) \Leftrightarrow$$

$$|\texttt{radius}(c_1) - \texttt{radius}(c_2)| \leq \texttt{dist}(\texttt{center}(c_1), \texttt{center}(c_2))$$

$$\wedge$$

$$\texttt{dist}(\texttt{center}(c_1), \texttt{center}(c_2)) \leq \texttt{radius}(c_1) + \texttt{radius}(c_2)$$

Example (2)

$$\forall\, c_1 : \texttt{circle},\; c_2 : \texttt{circle},\; x : \texttt{point}.$$

$$\left\{ \begin{array}{l} x\,\texttt{is-onc}\,c_1 \wedge \\ x\,\texttt{is-onc}\,c_2 \end{array} \right\}$$

$$\Leftrightarrow$$

$$\left\{ \begin{array}{l} \texttt{if } \delta_1(c_1, c_2) \\ \qquad \texttt{then } \mathit{list} = [\texttt{intercc1}(c_1, c_2), \texttt{intercc2}(c_1, c_2)] \\ \qquad\qquad \texttt{for } p \texttt{ in } \mathit{list} \texttt{ do } x = p \texttt{ done} \\ \qquad \texttt{else if } c_1 = c_2 \texttt{ then } x\,\texttt{is-onc}\,c_1 \\ \qquad\qquad\quad \texttt{else fail}\} \end{array} \right\}$$

Axioms system and inferences

Simple example of *ad-hoc* system of axioms

$\mathtt{dist}(X, Y) = \mathtt{dist}(Y, X)$
$\mathtt{mid}(X, Y) = \mathtt{mid}(Y, X)$
...
$X \ \mathtt{is\text{-}onl}\ Z \wedge Y \ \mathtt{is\text{-}onl}\ Z \wedge X \neq Y \supset Z = \mathtt{lpp}(X, Y)$
$Z = \mathtt{lpp}(X, Y) \supset X \ \mathtt{is\text{-}onl}\ Z \wedge Y \ \mathtt{is\text{-}onl}\ Z$
$O = \mathtt{center}(C) \wedge L = \mathtt{radius}(C \supset C = \mathtt{ccr}(O, L)$
$C = \mathtt{ccr}(O, L) \supset L = \mathtt{radius}(C) \wedge O = \mathtt{center}(C)$
$X \ \mathtt{is\text{-}onl}\ D_1 \wedge X \ \mathtt{is\text{-}onl}\ D_2 \wedge D_1 \neq D_2 \supset X = \mathtt{interll}(D_1, D_2)$
$X = \mathtt{interll}(D_1, D_2) \supset X \ \mathtt{is\text{-}onl}\ D_1 \wedge X \ \mathtt{is\text{-}onl}\ D_2$
$\mathtt{iso}(A, B, C) \supset B \neq C$
...
$\mathtt{dist}(A, B) = K \supset B \ \mathtt{is\text{-}onc}\ \mathtt{ccr}(A, K)$
$\mathtt{lpp}(A, B) \ \mathtt{ortho}\ \mathtt{lpp}(A, C) \wedge B \neq C \supset A \ \mathtt{is\text{-}onc}\ \mathtt{cdiam}(B, C)$
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First order and a little bit more: a toy example

A toy axiomatic:

(A1) $\forall\, x,\ o,\ r\ (\mathrm{app}(x, \mathrm{ccr}(o, r)) \Leftrightarrow \mathrm{egd}(x, o, r))$

(A2) $\forall\, C_1,\ C_2\ \exists x\ (\mathrm{app}(x, C_1) \wedge \mathrm{app}(x, C_2))$

we want to prove:

(F) $\forall a \forall b \forall l_1 \forall l_2 \exists x \cdot (\mathrm{egd}(a, x, l_1) \wedge \mathrm{egd}(b, x, l_2))$

By refutation and applying Skolem's method, we have:

$$\neg\,\mathrm{egd}(a, X, l_1) \vee \neg\,\mathrm{egd}(b, X, l_2) \tag{1}$$

$$\neg\,\mathrm{app}(X, \mathrm{ccr}(O, R)) \vee \mathrm{egd}(X, O, R) \tag{2}$$

$$\mathrm{app}(X, \mathrm{ccr}(O, R)) \vee \neg\,\mathrm{egd}(X, O, R) \tag{3}$$

$$\mathrm{app}(i(C_1, C_2), C_1) \tag{4}$$

$$\mathrm{app}(i(C_1, C_2), C_2) \tag{5}$$

toy example (2): Prolog program

```
egd(O, X, R) :- app(X, ccr(O, R)).
app(i(C1, C2), C1).
app(i(C2, C1), C2).
app(X, ccr(O, R)) :- egd(O, X, R).
```

Goal:

```
?- egd(a, C, l1), egd(b, C, l2).
```

Prolog's answer:

```
C = i(ccr(a, l1), ccr(b, l2))
```

A step forward FO : the '$\texttt{known}$' predicate

The idea is to mimick Prolog behavior with more control.
For every sort α, we define the predicate $\texttt{known}$:
$\texttt{known} : \alpha \rightarrow$
($\texttt{known} : \alpha \rightarrow$ Prop, for the Coq addicts)

With the following axioms for every functionnal symbol
$f : s_1 \dots s_k \rightarrow s$:
$\forall x_1 : s_1 \dots \forall x_k : s_k,$
$\texttt{known}(x_1) \wedge \dots \texttt{known}(x_k) \supset \texttt{known}(f(x_1, \dots x_k))$

A problem with statement $\mathcal{C}(\mathcal{X}, \mathcal{A})$ where $\mathcal{X}$ are the unknowns and $\mathcal{A}$ the parameters, is put under the form:
Prove that: $\texttt{known}(\mathcal{A}) \wedge \mathcal{C}(\mathcal{X}, \mathcal{A}) \supset \texttt{known}(\mathcal{X})$

Utility of known

(Meta)-theorem

$$\left(\bigwedge_{i=1}^{n} \texttt{known}(a_i) \wedge C(\mathcal{X}, \mathcal{A}) \wedge \delta(\mathcal{X}, \mathcal{A}) \right) \supset \bigwedge_{j=1}^{m} \texttt{known}(x_j)$$

is a theorem in the considered geometric universe iff there are some terms such that:

$$C(\mathcal{X}, \mathcal{A}) \supset \left(\delta(\mathcal{X}, \mathcal{A}) \supset \bigvee_{l} X = F_l(\mathcal{A}) \right)$$

The first formulation can be used alongside a mechanism to keep book of equalities of terms (just like Prolog's mechanism). The preconditions are used to determine the validity domains $\delta(\mathcal{X}, \mathcal{A})$

A Prolog implementation

System of axioms

Axioms of different kinds

- for permutations
- for representation
- for proofs
- for construction

Different "inference" kinds

- Unification *modulo*
- proof of preconditions (guard)
- forward chaining for building a construction program

Low level axioms and control (signature)

Geometric sort: `point`

- degree of freedom: 2
- basic constructors: `interll`, `intercc`, ...
- automatic objects: no

functional symbol: `lpp`

- profile: `pointpoint`→ `line`
- decomposability: `(is-onl`, `is-onl)`
- equivalents: no
- preconditions: `except(lpp(A,B), A eg B)`

predicative symbol: `iso`

- profile: `point point point`
- equivalents: `iso(A,B,C) equiv iso(A,C,B)`

Unification

Unification modulo permutations

Example: Using rule `mid(A,B) equiv mid(B,A)`, the unification `mid(X,Y) = mid(a,b)` gives two unifiers: $X = a \wedge Y = b$ and $X = b \wedge Y = a$.

Unification modulo incidence relation

Example: if points A_1, A_2, A_3, A_4 are on line L, then L can be unified with, for instance, term $\texttt{lpp}(A_4, A_2)$ (use of basic constructors and decomposability notions).

Geometric proofs (1)

Geometric reasoning

Example : $\text{iso(A,B,C)} \supset \text{dist(A,B)} = \text{dist(A,C)}$

$\supset$ A `is-onl` bis(B, C)

$\supset$ A `is-onl` lortho(B,C,`mid`(B,C)) ...

Geometric proofs (2)

Proof and disjunction

When the solver has to apply a rule corresponding to the axiom:

$X \texttt{ is-onl } Z \wedge Y \texttt{ is-onl } Z \wedge X \neq Y \supset Z = \texttt{lpp}(X,Y)$

It has to prove that either $X = Y$ or $X \neq Y$. A small rule-based prover is used with rules dealing with dis-equalities like this one: $iso(A, B, C) \supset B \neq C$.

If it is able to prove

- $X = Y$, then the rule is not applied, but the information $X = Y$ is now usable,
- $X \neq Y$ then the rule is applied (and the dis-equality is put into a database).

If it cannot prove one of these two cases, both are taken into account and a "`if then else`" structure is used in the program to be built.

Standard rules (high level axioms)

Example of a no-constructive rule

```
if [iso(A,B,C)] then
  dist(A,B) '=l=' dist(A,C)
```

Example of a constructive rule

```
if [dist(A,B) '=l=' K] and
   [known A, known K, unknown B]
then [B is-onc ccr(A,K)].
```

The pseudo-logical `unkown` predicate is used to control the application of constructive rules.

Disjunctive rules

Example, the bissector rule:

```
 if  [did(A,D1) '=l=' did(A,D2)]
       and
[differents [D1,D2], known D1, known D2, unknown A]
      then
    either [dird(D1) diff dird(D2)]
                   and [ A is_onl bis(D1,D2) : 1]
       or
    either [dird(D1) eg dird(D2), D1 diff D2]
                    and [A is_onl dmd(D1,D2) : 1]
       or
    either [D1 eg D2] and [].
```

Use of a disjunctive rule

A dedicated prover tries to prove that one of the sub-conditions is true (for instance D1 and D2 are parallel) if it succeeds then the rule is applied with the corresponding conclusion
If not, an "if ...then...else" or a "switch ... case" structure is inserted in the program construction and all the cases are examined.

Conclusion

I roughly described a FO implementation of geometric constructions as I remember it, and I feel there is already interesting things to do in the domain ;-)

Хвала на пажњи