LF$_P$ – A Logical Framework with External Predicates

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in collaboration with
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Briefly about LF

The Harper-Honsell-Plotkin Logical Framework

- LF – a logical framework based on the $\lambda\Pi$-calculus
- Dependent types - types depending on terms
- Based on the Curry-Howard isomorphism
- Basis for the proof assistant Twelf
The main ideas

- Develop a way of easily and smoothly encoding logics with arbitrary structural side-conditions in LF,
- Separate derivation from verification/computation,
- Increase modularity, and
- Optimize performance.
The pseudo-syntax of LF$_P$

\[ \Sigma \in S \quad \Sigma ::= \emptyset | \Sigma, a:K | \Sigma, c:\sigma \quad \text{Signatures} \]
\[ \Gamma \in \mathcal{C} \quad \Gamma ::= \emptyset | \Gamma, \chi:\sigma \quad \text{Contexts} \]
\[ K \in \mathcal{K} \quad K ::= \text{Type} | \Pi \chi:\sigma.\ K \quad \text{Kinds} \]
\[ \sigma, \tau, \rho \in \mathcal{F} \quad \sigma ::= a | \Pi \chi:\sigma.\tau | \sigma \ N \ | \mathcal{L}_{N,\sigma}^P[\rho] \quad \text{Families} \]
\[ M, N \in \mathcal{O} \quad M ::= c | x | \lambda \chi:\sigma.\ M \ | \ M \ N \ | \mathcal{L}_{N,\sigma}^P[M] \ | \mathcal{U}_{N,\sigma}^P[M] \quad \text{Objects} \]

Figure: The pseudo-syntax of LF$_P$
So, what is new?

- Predicates on derivable typing judgements $\mathcal{P}(\Gamma \vdash_{\Sigma} N : \sigma)$
  - Truth verified via an external call to a logical system,
  - Can inspect the signature, context, term, and the type.
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- Introduction rules:

\[
\frac{\Gamma \vdash_{\Sigma} \rho : \text{Type} \quad \Gamma \vdash_{\Sigma} N : \sigma}{\Gamma \vdash_{\Sigma} L^P_{\sigma}[\rho] : \text{Type}} \quad \frac{\Gamma \vdash_{\Sigma} M : \rho \quad \Gamma \vdash_{\Sigma} N : \sigma}{\Gamma \vdash_{\Sigma} L^P_{\sigma}[M] : L^P_{\sigma}[\rho]}
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- Elimination rule:
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  \frac{\Gamma \vdash \Sigma M : \mathcal{L}_{N,\sigma}^\mathcal{P}\[\rho] \quad \Gamma \vdash \Sigma N : \sigma \quad \mathcal{P}(\Gamma \vdash \Sigma N : \sigma)}{\Gamma \vdash \Sigma \mathcal{U}_{N,\sigma}^\mathcal{P}\[M] : \rho}\]
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- $\mathcal{L}$-reduction: $\mathcal{U}_{N,\sigma}[\mathcal{L}_{N,\sigma}[M]] \rightarrow_{\mathcal{L}} M$. 
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The main properties

- Confluence, Strong Normalization - Yes, immediately.
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- Subject Reduction - Yes, with certain conditions imposed on predicates (closure under signature and context weakening and permutation, substitution, and $\beta\mathcal{L}$-reduction.

Decidability - Yes, if predicates used are decidable.

A canonical version of the system ($\text{LF}_C$) was also developed, dealing only with $\beta\eta$-long normal forms, for easy formulation and proofs of adequacy of the encodings.
## Properties of LFₚ

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- **Decidability** - Yes, if predicates used are decidable.
- **A canonical version of the system** ($\text{LF}_\mathcal{P}^c$) was also developed, dealing only with $\beta\eta$-long normal forms, for easy formulation and proofs of adequacy of the encodings.
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- Non-commutative linear logic
  - Conditions on occurrence and order of assumptions
- A simple imperative language with Hoare-like logic
  - Pre- and post-conditions
Thank you for your attention!
Any questions?