Automated Solving of Triangle Construction Problems

— ongoing work —

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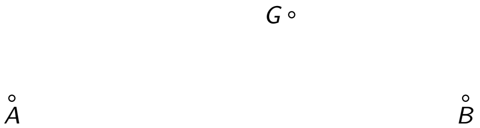
Fifth Workshop on Formal and Automated Theorem Proving and Applications
Belgrade, February 3-4, 2012.

Constructions with Straightedge and Compass

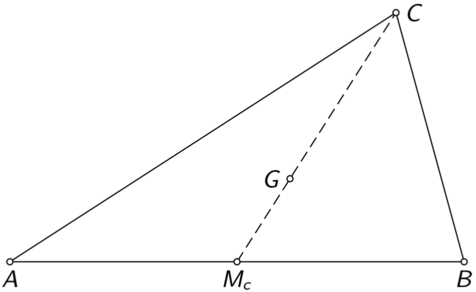
- Goal: construct a triangle that meets given constraints
- Widely studied on all education levels
- Main obstacle: combinatorial explosion — huge search space:
  - many different construction steps
  - plenty of objects that each step could be applied to
- The construction has to be accompanied by a proof that it meets the given specification

Example Problem



Problem: *Construct a triangle ABC given vertices A and B and the barycenter G*

Problem Solution



Solution: *Construct the midpoint* M_c *of the segment* AB*; then construct the vertex* C *such that* $M_cG : M_cC = 1/3$

Existing Approaches

- Just a couple of existing approaches, including:
  - Gao and Chou (1998)
  - Schreck (2001)
  - Gulwani et.al (2011)

Wernick's Problems

- Created in 1982, some variants in the meanwhile
- Task: construct a triangle given three located points selected from the following list:
  - A, B, C – vertices
  - I, O – incenter and circumcenter
  - H, G – orthocenter and barycenter
  - M_a, M_b, M_c – the side midpoints
  - H_a, H_b, H_c – feet of vertices on the opposite sides
  - T_a, T_b, T_c – intersections of the internal angles bisectors with the opposite sides

Wernick's Problems (2)

139 non-trivial, significantly different, problems; 25 redundant (R) or locus-restricted (L); some solvable (S), some unsolvable (U); 15 still with unknown status

1. A, B, O L	[illegible]	57. A, H, I S [9]	85. M_a, M_b, H_a S	113. M_a, T_b, T_c
2. A, B, M_a S	[illegible]	A, T_a, T_b S [9]	86. M_a, M_b, H_c S	114. M_a, T_b, I U [9]
3. A, B, M_c R	[illegible]	T_a, I L	87. M_a, M_b, H S [9]	115. G, H_a, H_b U [9]
4. A, B, G S	[illegible]	T_b, T_c S	88. M_a, M_b, T_a U [9]	116. G, H_a, H S
5. A, B, H_a L	[illegible]	I S	89. M_a, M_b, T_c U [9]	117. G, H_a, T_a S
6. A, B, H_c L	[illegible]	M_b S	90. M_a, M_b, I U [10]	118. G, H_a, T_b
7. A, B, H S	[illegible]	G S	91. M_a, G, H_a L	119. G, H_a, I
8. A, B, T_a S	[illegible]	H_a L	92. M_a, G, H_b S	120. G, H, T_a U [9]
9. A, B, T_c L	[illegible]	S	93. M_a, G, H S	121. G, H, I U [9]
[illegible]	[illegible]	S	94. M_a, G, T_a S	122. G, T_a, T_b
[illegible]	[illegible]	L	95. M_a, G, T_b U [9]	123. G, T_a, I
[illegible]	[illegible]	U [9]	96. M_a, G, I S [9]	124. H_a, H_b, H_c S
[illegible]	[illegible]	S	97. M_a, H_a, H_b S	125. H_a, H_b, H S
[illegible]	[illegible]	S	98. M_a, H_a, H L	126. H_a, H_b, T_a S
[illegible]	[illegible]	R	99. M_a, H_a, T_a L	127. H_a, H_b, T_c
[illegible]	[illegible]	U [9]	100. M_a, H_a, T_b U [9]	128. H_a, H_b, I
[illegible]	[illegible]	U [9]	101. M_a, H_a, I S	129. H_a, H, T_a L
[illegible]	[illegible]	H_b U [9]	102. M_a, H_b, H_c L	130. H_a, H, T_b U [9]
[illegible]	[illegible]	H S	103. M_a, H_b, H S	131. H_a, H, I S [9]
[illegible]	[illegible]	T_a S	104. M_a, H_b, T_a S	132. H_a, T_a, T_b
[illegible]	[illegible]	H_a, T_b	105. M_a, H_b, T_b S	133. H_a, T_a, I S
[illegible]	[illegible]	H_a, I	106. M_a, H_b, T_c U [9]	134. H_a, T_b, T_c
[illegible]	[illegible]	O, H, T_a U [9]	107. M_a, H_b, I U [9]	135. H_a, T_b, I
24. [illegible]	[illegible]	80. O, H, I U [9]	108. M_a, H, T_a U [9]	136. H, T_a, T_b
25. $A,$ [illegible]	[illegible]	81. O, T_a, T_b	109. M_a, H, T_b U [10]	137. H, T_a, I
26. $A, M_a,$ [illegible]	T_b, I S	82. O, T_a, I S [9]	110. M_a, H, I U [10]	138. T_a, T_b, T_c U [11]
27. A, M_a, I S [9]	55. A, H, T_a S	83. M_a, M_b, M_c S	111. M_a, T_a, T_b U [10]	139. T_a, T_b, I S
28. A, M_b, M_c S	56. A, H, T_b U [9]	84. M_a, M_b, G S	112. M_a, T_a, I S	

Basic Approach (1)

- Following careful analysis of all solutions
- Constructions consist of high-level construction steps (for example: *if barycenter G and circumcenter O are known, then the orthocenter H can be constructed*)
- Simple forward chaining mechanism for search procedure
- Points - only basic objects; lines and circles defined as functions of their points
- Implemented in Prolog

Basic Approach (2)

- Around 70 general rules used
- Example: *if two triangle vertices are given, then the side bisector can be constructed*
- For symmetric predicates, no redundant facts are derived
- Solves 60 examples from Wernick's list, each in less than 1s and with the maximal search depth 9
- But... there are too many rules! (it is not problem to search over them, but to invent them)

Separation of concepts – definitions, lemmas, construction steps (1)

Motivating example: *Construct the midpoint M_c of AB and then construct C such that $M_cG : M_cC = 1 : 3$* uses the facts:

- M_c is the side midpoint of AB
- G is the barycenter of ABC
- it holds that $M_cG = 1/3M_cC$
- given points X and Y, it is possible to construct the midpoint of the segment XY
- given points X and Y, it is possible to construct a point Z, such that: $XY : XZ = 1 : 3$

Separation of concepts – definitions, lemmas, construction steps (2)

Motivating example: *Construct the midpoint M_c of AB and then construct C such that $M_cG : M_cC = 1 : 3$* uses the facts:

- M_c is the side midpoint of AB (definition of M_c)
- G is the barycenter of ABC (definition of G)
- it holds that $M_cG = 1/3M_cC$ (lemma)
- given points X and Y, it is possible to construct the midpoint of the segment XY (construction primitive)
- given points X and Y, it is possible to construct a point Z, such that: $XY : XZ = 1 : 3$ (construction primitive)

Advanced Approach

- Task: Derive high-level (instantiated) construction steps from the set of definitions, lemmas and construction primitives
- From:
  - it holds that $M_c G = 1/3 M_c C$ (lemma)
  - given points X and Y, it is possible to construct a point Z, such that: $XY : XZ = 1 : r$ (construction primitive)

  we can derive:
  - given M_c and G, it is possible to construct C

Rule derivation

- Limit instantiations of definitions/lemmas
- So far, half of the rules of the basic system are derived from:
  - around 15 definitions (including Wernick's notation)
  - around 10 lemmas
  - only 2 suitable construction primitives
- Deriving rules is performed once, in preprocessing phase (takes approx. 20s)

Discussion

- Objection: the approach is problem-tailored!
  - Answer: no system can invent all needed lemmas, so other systems are too problem-tailored
- Objection: how can the approach be used for other families of problems?
  - Answer: in analogy with this family (the knowledge may overlap partly)
- Objection: ...then, it might become inefficient?
  - Answer: It could automatically choose over domains

Future Work

- Complete the process of automated deriving of rules
- Automated generation of constructions and figures in GCLC (along with a construction description in LaTeX)
- Proving (in GCLC) that the constructions meet specifications, using automated theorem provers
- Proving (in coherent logic, by ArgoCLP prover) that constructed points indeed exist (under some conditions)