Incompletely Specified Operations and their Clones

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Progress in Decision Procedures: From Formalizations to Applications

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Previously on this subject...

Jelena Čolić, Hajime Machida and Jovanka Pantović.
Clones of Incompletely Specified Operations.
ISMVL 2012, pages 256–261.

Jelena Čolić, Hajime Machida and Jovanka Pantović.
One-point Extension of the Algebra of Incompletely Specified Operations.
Multiple-Valued Logic and Soft Computing,
to be published in 2013.

Jelena Čolić.
On the Lattice of Clones of Incompletely Specified Operations.
Conference on Universal Algebra and Lattice Theory, Szeged 2012.

What is an IS operation?

Total operation:

OR	0	1
0	0	1
1	1	1

Let

$$h(x_1, x_2) = \mathtt{OR}(g(x_1), x_2)$$

Partial operation:

$$\mathtt{OR}(g(x_1), 1) \text{ undefined if } g(x_1) \text{ is undefined}$$

Incompletely specified operation:

$$\mathtt{OR}(g(x_1), 1) \quad = \quad 1$$

How to define it formaly?

Let A be a finite set and $k \notin A$.
Partial operation:

$$f : A^n \to A \cup \{k\}, \qquad k - \text{undefined}$$

Incompletely specified operation:

$$f : A^n \to A \cup \{k\}, \qquad k - \text{unspecified}$$

I_A - set of all IS operations on A

Composition of total and hyperoperations

- The composition of $f \in O_A^{(n)}$ and $g_1, \ldots, g_n \in O_A^{(m)}$ is an m-ary operation defined by

$$f(g_1, \ldots, g_n)(x_1, \ldots, x_m) = f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m)).$$

- The composition of $f \in H_A^{(n)}$ and $g_1, \ldots, g_n \in H_A^{(m)}$ is an m-ary hyperoperation defined by

$$f(g_1, \ldots, g_n)(x_1, \ldots, x_m) = \bigcup_{\substack{(y_1, \ldots, y_n) \in A^n \\ y_i \in g_i(x_1, \ldots, x_m) \\ 1 \leq i \leq n}} f(y_1, \ldots, y_n)$$

New composition

Definition

Let $f \in I_A^{(n)}$ and $g_1, \ldots, g_n \in I_A^{(m)}$. The i-composition of f and $g_1, \ldots, g_n$ is an m-ary IS operation defined by

$$f[g_1, \ldots, g_n](x_1, \ldots, x_m) = \prod_{\substack{(y_1, \ldots, y_n) \in A^n \\ y_i \sqsubseteq g_i(x_1, \ldots, x_m) \\ 1 \leq i \leq n}} f(y_1, \ldots, y_n)$$

where

$$\prod\{x_i : 1 \leq i \leq l\} = \begin{cases} x_1 & \text{, if } \; x_1 = x_2 = \ldots = x_l, \\ k & \text{, otherwise.} \end{cases}$$

$$\sqsubseteq = \{(x, x) : \; x \in A \cup \{k\}\} \cup \{(x, k) : \; x \in A\}$$

Example

$A = \{0, 1\}$
composition of partial operations

$$\begin{array}{c|cc} \mathrm{OR} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array} \qquad \begin{array}{c|c|c} & g_1 & g_2 \\ \hline 0 & 1 & 2 \\ 1 & 0 & 0 \end{array} \qquad \Rightarrow \qquad \begin{array}{c|c} & \mathrm{OR}(g_1, g_2) \\ \hline 0 & 2 \\ 1 & 0 \end{array}$$

$\mathrm{OR}(g_1, g_2)(0) = \mathrm{OR}(g_1(0), g_2(0)) = 2$

i-composition of IS operations

$$\begin{array}{c|cc} \mathrm{OR} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array} \qquad \begin{array}{c|c|c} & g_1 & g_2 \\ \hline 0 & 1 & 2 \\ 1 & 0 & 0 \end{array} \qquad \Rightarrow \qquad \begin{array}{c|c} & \mathrm{OR}[g_1, g_2] \\ \hline 0 & 1 \\ 1 & 0 \end{array}$$

$\mathrm{OR}[g_1, g_2](0) = \mathrm{OR}(g_1(0), g_2(0)) = \mathrm{OR}(1, 0) \sqcap \mathrm{OR}(1, 1) = 1 \sqcap 1 = 1$

IS clone

- $e_i^{n,A}(x_1, \ldots, x_i, \ldots, x_n) = x_i$ is an i-th n-ary projection.

Definition

A set $C \subseteq I_A$ is called a clone of incompletely specified operations (or IS clone) if

- *C contains all projections and*
- *C is closed with respect to i-composition.*

IS clone (second definiton)

- for $f \in I_A^{(1)}$ let $\zeta f = \tau f = \Delta f = f$;
- for $f \in I_A^{(n)}, n \geq 2$, let $\zeta f, \tau f \in I_A^{(n)}$ and $\Delta f \in I_A^{(n-1)}$ be defined as
  - $(\zeta f)(x_1, x_2, \dots, x_n) = f(x_2, \dots, x_n, x_1)$
  - $(\tau f)(x_1, x_2, x_3, \dots, x_n) = f(x_2, x_1, x_3, \dots, x_n)$
  - $(\Delta f)(x_1, x_2, \dots, x_{n-1}) = f(x_1, x_1, x_2 \dots, x_{n-1})$
- for $f \in I_A^{(n)}$ and $g \in I_A^{(m)}$ let $f \diamond g \in I_A^{(m+n-1)}$ be defined as

$$(f \diamond g)(x_1, \dots, x_{m+n-1}) = \prod_{\substack{y \in A \\ y \sqsubseteq g(x_1, \dots, x_m)}} f(y, x_{m+1}, \dots, x_{m+n-1})$$

Example

$A = \{0, 1\}$

OR	0	1
0	0	1
1	1	1

	g
0	0
1	2

Let $h(x_1, x_2) = \mathrm{OR}(g(x_1), x_2)$.

For partial operations:

h	0	1
0	0	1
1	2	2

$h(1, 1) = \mathrm{OR}(g(1), 1) = 2$

For IS operations:

h	0	1
0	0	1
1	2	1

$h(1, 1) = \mathrm{OR}(g(1), 1) = \mathrm{OR}(0, 1) \sqcap \mathrm{OR}(1, 1) = 1 \sqcap 1 = 1$

IS clone (second definiton)

$\mathcal{I}_A = (I_A; \diamond, \zeta, \tau, \Delta, e_1^{2,A})$ full algebra of IS operations

Theorem

$C \subseteq I_A$ is an IS clone if and only if C is a subuniverse of the full algebra of IS operations.

IS operations vs. hyperoperations

$\lambda : H_A \to I_A$, $f \mapsto f^{is}$

$$f^{is}(x_1, \ldots, x_n) = \begin{cases} f(x_1, \ldots, x_n) & , |f(x_1, \ldots, x_n)| = 1 \\ k & , \text{otherwise} \end{cases}$$

Theorem

(i) *For* $|A| = 2$, λ *is an isomorphism from* $(H_A; \circ, \zeta, \tau, \Delta, e_1^{2,A})$ *to* $(I_A; \diamond, \zeta, \tau, \Delta, e_1^{2,A})$.

(ii) *For* $|A| \geq 3$, λ *is a homomorphism from* $(H_A; \zeta, \tau, \Delta, e_1^{2,A})$ *to* $(I_A; \zeta, \tau, \Delta, e_1^{2,A})$.

(iii) *For* $|A| \geq 3$, *there exist* $f, g \in H_A$ *satisfying* $\lambda(f \circ g) \neq \lambda(f) \diamond \lambda(g)$.

Example

$A = \{0, 1, 2\}$

(ii) λ is not injective

	f
0	$\{0\}$
1	$\{1\}$
2	$\{0, 1\}$

	g
0	$\{0\}$
1	$\{1\}$
2	$\{0, 2\}$

$\Rightarrow$

	$f^{is} = g^{is}$
0	0
1	1
2	3

Example

$A = \{0, 1, 2\}$

(iii) $(f \circ g)^{is} \neq f^{is} \diamond g^{is}$

f	0	1	2
0		$\{1\}$	
1		$\{0,1\}$	
2		$\{1\}$	

	g
0	$\{0\}$
1	$\{0\}$
2	$\{0,2\}$

$\Rightarrow$

$(f \circ g)^{is}$	0	1	2
0		1	
1		1	
2		1	

$f \circ g(2, 1) = f(0, 1) \cup f(2, 1) = \{1\}$

f^{is}	0	1	2
0		1	
1		3	
2		1	

	g^{is}
0	0
1	0
2	3

$\Rightarrow$

$f^{is} \diamond g^{is}$	0	1	2
0		1	
1		1	
2		3	

$f^{is} \diamond g^{is}(2, 1) = f^{is}(0, 1) \sqcap f^{is}(1, 1) \sqcap f^{is}(2, 1) = 3$

One-point extension

Let us define the mapping $I_A \to O_{A\cup\{k\}} : f \mapsto f^+$, as follows:

$$f^+(x_1,\dots,x_n) = \prod_{\substack{(y_1,\dots,y_n)\in A^n,\\ (y_1,\dots,y_n)\sqsubseteq(x_1,\,\dots,\,x_m)}} f(y_1,\dots,y_n)$$

$F^+ = \{f^+ : f \in F\} \subseteq O_{A\cup\{k\}}$ for $F \subseteq I_A$.

Example (one-point extension)

$A = \{0, 1\}$
Partial operation:

OR^+	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\mathrm{OR}^+(2, 1) = 2$
Incompletely specified operation:

OR^+	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

$\mathrm{OR}^+(2, 1) = \mathrm{OR}(0, 1) \sqcap \mathrm{OR}(1, 1) = 1 \sqcap 1 = 1$

Algebra of extended IS operations

Mapping $I_A \to I_A^+ : f \mapsto f^+$, is not an isomorphism from $(I_A; \diamond, \zeta, \tau, \Delta, e_1^{2,A})$ to $(I_A^+; \circ, \zeta, \tau, \Delta, e_1^{2,A\cup\{k\}})$.

I_A^+ is closed w.r.t. ζ, τ and $e_1^{2,A\cup\{k\}}$:

- $\left(e_1^{2,A}\right)^+ = e_1^{2,A\cup\{k\}}$
- $(\zeta f)^+ = \zeta(f^+)$
- $(\tau f)^+ = \tau(f^+)$

I_A^+ is not closed w.r.t. Δ and $\circ$:

- $(\Delta f)^+ \neq \Delta(f^+)$
- $(f \diamond g)^+ \neq f^+ \circ g^+$

Example

$(\Delta f)^+ \neq \Delta(f^+)$

f	0	1
0	0	0
1	2	0

$\Rightarrow$

	Δf
0	0
1	0

$\Rightarrow$

	$(\Delta f)^+$
0	0
1	0
2	0

f^+	0	1	2
0	0	0	0
1	2	0	2
2	2	0	2

$\Rightarrow$

	$\Delta(f^+)$
0	0
1	0
2	2

Example

$|A| \geq 3 \Rightarrow (f \diamond g)^+ \neq f^+ \circ g^+$

f	0	1	2
0			1
1			1
2			2

	g
0	0
1	0
2	1

$\Rightarrow$

$(f \diamond g)^+$	0	1	2	3
0			1	
1			1	
2			1	
3			1	

$$\begin{aligned} (f \diamond g)^+(3,2) &= f(g(0),2) \sqcap f(g(1),2) \sqcap f(g(2),2) \\ &= f(0,2) \sqcap f(0,2) \sqcap f(1,2) = 1 \end{aligned}$$

f^+	0	1	2	3
0			1	
1			1	
2			2	
3			3	

	g^+
0	0
1	0
2	1
3	3

$\Rightarrow$

$f^+ \circ g^+$	0	1	2	3
0			1	
1			1	
2			1	
3			3	

$(f^+ \circ g^+)(3,2) = f^+(g^+(3),2) = 3$

Extended IS clones

Full algebra of extended IS operations:

$$\mathcal{I}_A^+ = (I_A^+; \circ_i, \zeta, \tau, \Delta_i, e_1^{2, A\cup\{k\}})$$

where

$$\Delta_i(f) = (\Delta f)^+$$

$$f \circ_i g = (f \diamond g)^+$$

Theorem

$C \subseteq I_A^+$ is an extended IS clone
iff
C is a subuniverse of the full algebra $\mathcal{I}_A^+$ of extended IS operations.

Future work

- Further investigation of the lattice of IS clones:
  - maximal IS clones
  - minimal IS clones
- Describing all IS clones using relations
- Possible connection between IS clones and CSP

Thank you for your attention!