A syntax approach to automated detection of some redundancies in linear logic sequent derivations

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Example 1. Successful proof.

$$\dfrac{\dfrac{}{t \vdash t}\,Ax \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{r \vdash r}\,Ax}{r \vdash ?p, r}\,w?R}{r \vdash ?q, ?p, r}\,w?R}{r \vdash ?q \wp ?p, r}\,\wp R}{r \vdash (?q \wp ?p) \oplus s, r}\,\oplus R}{r, t \vdash t \otimes ((?q \wp ?p) \oplus s), r}\,\otimes R$$

- **the core (skeleton) of the proof?**
- **not used in Axioms:** p, q, **and** s
- **template for** $(3^2 - 1) \cdot 3$ **proofs**

$$\boldsymbol{r, t \vdash (t \otimes (?Q \wp ?F)) \oplus S,\ r}$$
$$\boldsymbol{r, t \vdash (t \otimes (?Q \wp ?F)),\ r}$$
$$\boldsymbol{r, t \vdash (t \otimes ?F) \oplus S,\ r}$$
$$\boldsymbol{r, t \vdash (t \otimes ?F), r}$$

$$\dfrac{\dfrac{}{\boldsymbol{t \vdash t}}\,\boldsymbol{Ax} \qquad \dfrac{\dfrac{}{\boldsymbol{r \vdash r}}\,\boldsymbol{Ax}}{\boldsymbol{r \vdash ?F, r}}\,\boldsymbol{w?R}}{\boldsymbol{r, t \vdash t \otimes ?F, r}}\,\boldsymbol{\otimes R}$$

Example 2. Formulae distribution

$$\dfrac{? \vdash q, ? \qquad ? \vdash r, ?}{\Gamma_1, \Gamma_2 \ \vdash \ q \otimes r, \Delta_1, \Delta_2} \otimes R$$

Not used as principal formula: $r \wp p$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{q, r \wp p \vdash q}\, Ax}{q, r \wp p \vdash q, ?s}\, w?R}{q, r \wp p \vdash q, ?s \oplus p}\, \oplus R \qquad r \wp p \vdash r}{q, r \wp p \vdash q \otimes r, ?s \oplus p} \otimes R$$

Redundant: $?s \oplus p$

$$\dfrac{\dfrac{\dfrac{\dfrac{\boldsymbol{r \wp p}}{q, r \wp p \vdash q}\, Ax}{q, r \wp p \vdash q, ?s}\, w?R}{q, r \wp p \vdash q, ?s \oplus p}\, \oplus R \qquad \dfrac{\dfrac{}{r \vdash r}\, Ax \qquad \dfrac{\dfrac{}{p \vdash p}\, Ax}{p \vdash ?s \oplus p}\, \oplus R}{r \wp p \vdash r, ?s \oplus p}\, \wp L}{q, r \wp p \vdash q \otimes r, ?s \oplus p} \otimes R$$

Example 3. Failed proof attempt.

$$\dfrac{\dfrac{}{r \vdash r}\,Ax \qquad p, {\color{green}s} \overset{?}{\vdash} p, {\color{green}q}}{p, r, s \vdash r \otimes p, q}\,\otimes R$$

$$\dfrac{}{r \vdash r}\,Ax \qquad\qquad \dfrac{\dfrac{}{r \vdash r}\,Ax \qquad \dfrac{}{p \vdash p}\,Ax}{p, r \vdash r \otimes p}\,\otimes R$$

- **Linear Logic**

  – **Logic of resources (controlled weakening and contraction)**

  linear formula ϕ

  exponential formula $?\phi$ **or** $!\phi$

  – **Non-monotonic logic** $\Gamma \vdash \Delta$

  $\Gamma, \phi \vdash \Delta \quad \Gamma \vdash \psi, \Delta \quad \Gamma, \phi \vdash \psi, \Delta$

  $$p, q \vdash p, r, s$$

  – **Can be used as a meta-logic**

- **Our view of redundant (sub)formula:**

  1) neither used in axioms and initial rules nor critical for enabling proof branching

  2) elimination which does not alter the search strategy applied (i.e. rule instances may be deleted);

  3) no additional proof search is required (i.e. rule instances may not be added);

  4) no loss of information at the leaves (i.e. axioms and some initial rules remain unchanged).

- Various approaches for detecting whether or not a formula occurrence is actually used in a derivation.
- **Labelling and constraints techniques**

$$\phi_1, \phi_2, \ldots \phi_k \quad \vdash \quad \psi_1, \psi_2, \ldots \psi_n$$

$$\phi_{1,[v_1]}, \phi_{2,[v_2]}, \ldots \phi_{k,[v_k]} \vdash \psi_{1,[w_1]}, \psi_{2,[w_2]}, \ldots \psi_{n,[w_n]} - \mathcal{C} \Updownarrow \mathcal{L}$$

- **Accumulation of constraints during proof-search:**

$$\frac{Premise(s) \qquad - \ \mathcal{C} \cup \{constraint(s)\} \Updownarrow \mathcal{L}}{Conclusion \qquad - \ \mathcal{C} \Updownarrow \mathcal{L}}$$

- **General form of constraints:**

$$\overline{\Gamma} = 1$$

$$\overline{p} = 0$$

$$\overline{\Gamma} \leq \overline{\Delta}$$

$$\langle\, \overline{F_{[wu_1]}} \rangle \;=\; \langle\, \overline{F_{[wu_2]}} \rangle$$

- **Boolean expressions**

formula $F_{[w]}$	corresponding Boolean expression $\overline{F_{[w]}}$
$F_{[w]} = (?(p \otimes ?q) \wp p_1)_{[w]}$	$\overline{F_{[w]}} = p_{[w]} + q_{[w]} + p_{1,[w]}$
$F_{[w]} = (?p \otimes (?q \wp p_1))_{[w]}$	$\overline{F_{[w]}} = p_{[w]} + q_{[w]} + p_{1,[w]}$
$F_{[w]} = p_{5,[w]}$	$\overline{F_{[w]}} = p_{5,[w]}$
$F_{[w]} = ?(\top_j \oplus ?p_i)_{[w]}$	$\overline{F_{[w]}} = c\top_{j,[w]} + p_{i,[w]}$

- **EXAMPLES OF RULES:**

- Axioms and initial rules;

$$\frac{-\mathcal{C} \cup \{p_i = 1, p_j = 1\} \Updownarrow \mathcal{L} := \{p_i \vdash p_j\}}{p_i \vdash p_j \quad - \mathcal{C} \Updownarrow \mathcal{L}} Ax$$

- Rules that discharge some formulas

$$\frac{\Gamma \vdash \phi, \Delta \quad - \mathcal{C} \cup \{\overline{\psi} \leq \overline{\phi}\} \Updownarrow \mathcal{L}}{\Gamma \vdash \phi \oplus \psi, \Delta \quad - \mathcal{C} \Updownarrow \mathcal{L}} \oplus R$$

$$\frac{\Gamma \vdash \Delta \quad - \mathcal{C} \cup \{\overline{\phi} \leq \overline{\Gamma} + \overline{\Delta}\} \Updownarrow \mathcal{L}}{\Gamma \vdash ?\phi, \Delta \quad - \mathcal{C} \Updownarrow \mathcal{L}} w?R$$

$$\frac{\vdash}{\vdash \ ?F} w?R$$

- Multiplicative binary rules:

$$\frac{\Gamma_1 \vdash \phi, \Delta_1 - \mathcal{C} \cup \{\overline{\Gamma_1} + \overline{\Delta_1} \leq \overline{\phi}\} \Updownarrow \mathcal{L}_1 \quad \Gamma_2 \vdash \psi, \Delta_2 - \mathcal{C} \cup \{\overline{\Gamma_2} + \overline{\Delta_2} \leq \overline{\psi}\} \Updownarrow \mathcal{L}_2}{\Gamma_1, \Gamma_2 \vdash \phi \otimes \psi, \Delta_1, \Delta_2 - \mathcal{C} \Updownarrow \mathcal{L}_1 \cup \mathcal{L}_2} \otimes$$

- Contraction rules:

$$\frac{\Gamma \vdash \ ?F_{[wx_1]},\ ?F_{[wx_2]},\ \Delta \quad - \mathcal{C} \cup \ C' \Updownarrow \mathcal{L}}{\Gamma \ \vdash \ ?F_{[w]},\ \Delta \quad - \mathcal{C} \Updownarrow \mathcal{L}} \ c?$$

where $\mathcal{C}' = \{ \ (a_1, a_2, \ldots a_n)_{[wx_1]} = (a_1, a_2, \ldots a_n)_{[wx_2]} \quad \vee$

$(a_1, a_2, \ldots a_n)_{[wx_1]} = (0, 0, \ldots 0) \quad \vee$

$(a_1, a_2, \ldots a_n)_{[wx_2]} = (0, 0, \ldots 0)$

$(a_1, a_2, \ldots a_n)_{[w]} = (a_1, a_2, \ldots a_n)_{[wx_1]} + (a_1, a_2, \ldots a_n)_{[wx_2]} \ \}$

- Additive binary rules;

- Rules that 'distribute' active formulae among the antecedent and succedent

- Rules with no additional constraints:

$$\frac{\Gamma, \phi, \psi \vdash \Delta \quad - \mathcal{C} \Updownarrow \mathcal{L}}{\Gamma, \phi \otimes \psi \vdash \Delta \quad - \mathcal{C} \Updownarrow \mathcal{L}} \otimes \mathbf{L}$$

Example: labelled proof search

$$
\dfrac{\Pi_1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{-C_{root} \cup \{r + r_1 \leq q + p + s,\ s \leq q + p,\ q \leq r + p + r_1,\ r = 1,\ r_1 = 1\} \ \Updownarrow \mathcal{L}_2 := \{r \vdash r_1\}}{r \vdash r_1 \ - \ C_{root} \cup \{q \leq r + p + r_1,\ r + r_1 \leq q + p + s,\ s \leq q + p,\ q \leq r + p + r_1\} \ \Updownarrow \mathcal{L}_2}\ w}{r \vdash ?p, r_1 \ - C_{root} \cup \{r + r_1 \leq q + p + s,\ s \leq q + p,\ q \leq r + p + r_1\} \ \Updownarrow \mathcal{L}_2}\ w?R}{r \ \vdash ?q, ?p, r_1 \ - C_{root} \cup \{r + r_1 \leq q + p + s,\ s \leq q + p\} \ \Updownarrow \mathcal{L}_2}\ \wp R}{r \ \vdash ?q \wp ?p, r_1 \ - C_{root} \cup \{r + r_1 \leq q + p + s,\ s \leq q + p\} \ \Updownarrow \mathcal{L}_2}\ \oplus R}{r \ \vdash \ (?q \wp ?p) \oplus s, r_1 \ - C_{root} \cup \{r + r_1 \leq q + p + s\} \ \Updownarrow \mathcal{L}_2}}{r, t \ \vdash \ (t_1 \otimes ((?q \wp ?p) \oplus s)), r_1 \quad - \ C_{root} \ \Updownarrow \mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2}\ \otimes R
$$

where Π_1 is

$$
\dfrac{- \quad C_{root} \cup \{t = 1,\ t_1 = 1, t \leq t_1\} \ \Updownarrow \mathcal{L}_1 := \{t \vdash t_1\}}{t \vdash t_1 \qquad - \ C_{root} \cup \{t \leq t_1\} \ \Updownarrow \mathcal{L}_1}\ Ax
$$

$$
C_{root} = \{r + t + t_1 + q + p + s = 1\}
$$

Algorithm RE *(input: labelled deduction $\boldsymbol{\pi}_l$)*

1. **Calculate possible assignments for the Boolean variables in $\boldsymbol{\pi}_l$;**
2. **If there is an assignment $\mathcal{I}$ with at least one Boolean variable being assigned the value 0, then:**
   - **2.1 Delete the atoms assigned 0 (i.e. delete the formulae made up of such atoms and the corresponding inferences).**
   - **2.2 Delete all labels and constraints.**

     EXIT: proof $\boldsymbol{\pi}'$.

   Else EXIT: 'Simplification of proof $\boldsymbol{\pi}_l$ is not possible'

Final set of constraints: $r + r_1 \leq q + p + s$, $s \leq q+p$, $r = 1$, $r_1 = 1$, $t_1 = 1$, $t = 1$, $t \leq t_1$

$\boxed{1.}$ *Calculate possible assignments:*

p	q	s
0	1	0
1	0	0
1	1	0
1	1	0
1	0	1

$\boxed{2.}$ *Eliminate redundant formulae:*

$$p = 0,\ q = 1,\ s = 0 \quad \mapsto \quad \dfrac{\dfrac{}{t \vdash t_1}\,Ax \qquad \dfrac{\dfrac{}{r \vdash r_1}\,Ax}{r \vdash ?q, r_1}\,?wR}{r, t \vdash t_1 \otimes ?q, r_1}\,\otimes R$$

$$p = 0,\ q = 1,\ s = 1 \quad \mapsto \quad \dfrac{\dfrac{}{t \vdash t_1}\,Ax \qquad \dfrac{\dfrac{\dfrac{}{r \vdash r_1}\,Ax}{r \vdash ?q, r_1}\,?wR}{r \vdash ?q \oplus s, r_1}\,\oplus R}{r, t \vdash t_1 \otimes (?q \oplus s), r_1}\,\otimes R$$

CONCLUSIONS and FUTURE WORK

- √ A technical, syntax approach;
- √ There was no a priori commitment to a particular search strategies;
- √ Sound and complete Algorithm RE;
- √ Contribution to a library of automated support tools for reasoning about sequent calculi proof search

- Analysis of failed proof attempts

$$\frac{-\ \mathcal{C} \cup \{p_j = 1, p_i = 1\} \ \Updownarrow \ \mathcal{L} := \{p_i \vdash p_j\}}{\Gamma,\ p_i \vdash p_j, \Delta \quad -\mathcal{C} \ \Updownarrow \ \mathcal{L}}\ qAx$$

- Complexity analysis

- Combine Algorithm RE with our procedure for loop detection during proof search