A syntax approach to automated detection of some redundancies in linear logic sequent derivations

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Example 1. Successful proof.

\[
\begin{align*}
\frac{r \vdash r}{r \vdash \text{Ax}} \\
\frac{r \vdash \text{Ax}}{r \vdash \neg p, r \vdash w?R} \\
\frac{r \vdash \neg q, ?p, r \vdash w?R}{r \vdash \neg q \neg p, r \vdash \varnothing R} \\
\frac{t \vdash t}{r, t \vdash t \otimes ((\neg q \neg p) \oplus s), r} \\
\frac{r \vdash (\neg q \neg p) \oplus s, r \vdash \varnothing R}{r, t \vdash t \otimes (\neg q \neg p) \oplus s, r \otimes R}
\end{align*}
\]

- the core (skeleton) of the proof?

- not used in Axioms: \( p, q, \) and \( s \)

- template for \((3^2 - 1) \cdot 3\) proofs

\[
\begin{align*}
r, t \vdash (t \otimes (\neg Q \neg F)) \oplus S, r \\
r, t \vdash (t \otimes (\neg Q \neg F)), r \\
r, t \vdash (t \otimes F) \oplus S, r \\
r, t \vdash (t \otimes F), r
\end{align*}
\]
Example 2. Formulae distribution

\[ \Gamma_1, \Gamma_2 \vdash q \otimes r, \Delta_1, \Delta_2 \otimes R \]

---

**Not used as principal formula:** *r \varnothing p*

\[ q, r \varnothing p \vdash q \quad \text{Ax} \]

\[ q, r \varnothing p \vdash q, ?s \quad w?R \]

\[ q, r \varnothing p \vdash q, ?s \oplus p \quad \text{Ax} \]

\[ q, r \varnothing p \vdash q \otimes r, ?s \oplus p \quad \otimes R \]

**Redundant: ** *? s \oplus p*

\[ q, r \varnothing p \vdash q \quad \text{Ax} \]

\[ q, r \varnothing p \vdash q, ?s \quad w?R \]

\[ q, r \varnothing p \vdash q, ?s \oplus p \quad \oplus R \]

\[ q, r \varnothing p \vdash q \otimes r, ?s \oplus p \quad \otimes R \]
Example 3. Failed proof attempt.

\[
\frac{r \vdash r \text{ Ax}}{p, r, s \vdash r \otimes p, q} \quad \frac{p, s \vdash p, q}{p, r, s \vdash r \otimes p, q} \quad \otimes R
\]

\[
\frac{r \vdash r \text{ Ax}}{p, r \vdash r \otimes p} \quad \frac{p \vdash p \text{ Ax}}{p, r \vdash r \otimes p} \quad \otimes R
\]
• **Linear Logic**

  - Logic of resources  (controlled weakening and contraction)
    
    linear formula \( \phi \)

    exponential formula \( ?\phi \) or \( !\phi \)

  - **Non-monotonic logic**
    
    \( \Gamma \vdash \Delta \)

    \( \Gamma, \phi \vdash \Delta \)

    \( \Gamma \vdash \psi, \Delta \)

    \( \Gamma, \phi \vdash \psi, \Delta \)

    \( p, q \vdash p, r, s \)

  - Can be used as a meta-logic
• Our view of redundant (sub)formula:

1) neither used in axioms and initial rules nor critical for enabling proof branching

2) elimination which does not alter the search strategy applied (i.e. rule instances may be deleted);

3) no additional proof search is required (i.e. rule instances may not be added);

4) no loss of information at the leaves (i.e. axioms and some initial rules remain unchanged ).
• Various approaches for detecting whether or not a formula occurrence is actually used in a derivation.

• Labelling and constraints techniques

\[ \phi_1, \phi_2, \ldots \phi_k \vdash \psi_1, \psi_2, \ldots \psi_n \]

\[ \phi_{1,[v_1]}, \phi_{2,[v_2]}, \ldots \phi_{k,[v_k]} \vdash \psi_{1,[w_1]}, \psi_{2,[w_2]}, \ldots \psi_{n,[w_n]} - \mathcal{C} \upharpoonright \mathcal{L} \]

• Accumulation of constraints during proof-search:

\[
\begin{array}{c}
\text{Premise}(s) \quad - \quad \mathcal{C} \cup \{\text{constraint}(s)\} \quad \upharpoonright \quad \mathcal{L} \\
\hline
\text{Conclusion} \quad - \quad \mathcal{C} \upharpoonright \mathcal{L}
\end{array}
\]
• General form of constraints:

\[ \bar{\Gamma} = 1 \]
\[ \bar{p} = 0 \]
\[ \bar{\Gamma} \leq \bar{\Delta} \]
\[ \langle \bar{F}_{[wu_1]} \rangle = \langle \bar{F}_{[wu_2]} \rangle \]

• Boolean expressions

<table>
<thead>
<tr>
<th>formula ( F_{[w]} )</th>
<th>corresponding Boolean expression ( \bar{F}_{[w]} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_{[w]} = (? (\pi ? q \circ p_1)_{[w]} )</td>
<td>( \bar{F}<em>{[w]} = p</em>{[w]} + q_{[w]} + p_{1,[w]} )</td>
</tr>
<tr>
<td>( F_{[w]} = (? p \otimes (\pi q \circ p_1))_{[w]} )</td>
<td>( \bar{F}<em>{[w]} = p</em>{[w]} + q_{[w]} + p_{1,[w]} )</td>
</tr>
<tr>
<td>( F_{[w]} = p_{5,[w]} )</td>
<td>( \bar{F}<em>{[w]} = p</em>{5,[w]} )</td>
</tr>
<tr>
<td>( F_{[w]} = ? (\top j \oplus ? p_i)_{[w]} )</td>
<td>( \bar{F}<em>{[w]} = c \top j</em>{[w]} + p_{i,[w]} )</td>
</tr>
</tbody>
</table>
- Axioms and initial rules:
  \[
  - \mathcal{C} \cup \{p_i = 1, p_j = 1\} \vdash \mathcal{L} := \{p_i \vdash p_j\} \\
  p_i \vdash p_j \quad - \quad \mathcal{C} \upharpoonright \downarrow \mathcal{L} \quad Ax
  \]

- Rules that discharge some formulas
  \[
  \Gamma \vdash \phi, \Delta \quad - \quad \mathcal{C} \cup \{\psi \leq \phi\} \vdash \mathcal{L} \quad \upharpoonright \downarrow \mathcal{L} \quad \oplus R
  \]

  \[
  \Gamma \vdash \Delta \quad - \quad \mathcal{C} \cup \{\overline{\phi} \leq \overline{\Gamma + \Delta}\} \vdash \mathcal{L} \quad \downharpoonright \downarrow \mathcal{L} \quad w?R
  \]

  \[
  \Gamma \vdash \psi, \Delta \quad - \quad \mathcal{C} \upharpoonright \downarrow \mathcal{L} \quad \vdash \psi \quad w?R
  \]

  \[
  \vdash \psi \quad w?R
  \]
- **Multiplicative binary rules:**

\[
\Gamma_1 \vdash \phi, \Delta_1 - C \cup \{\overline{\Gamma_1} + \overline{\Delta}_1 \leq \overline{\phi}\} \Downarrow L_1 \quad \Gamma_2 \vdash \psi, \Delta_2 - C \cup \{\overline{\Gamma_2} + \overline{\Delta}_2 \leq \overline{\psi}\} \Downarrow L_2
\]

\[
\Gamma_1, \Gamma_2 \vdash \phi \otimes \psi, \Delta_1, \Delta_2 - C \Downarrow L_1 \cup L_2
\]

- **Contraction rules:**

\[
\Gamma \vdash ?F_{[wx_1]}, \ ?F_{[wx_2]}, \ \Delta - C \cup C' \Downarrow L \quad \text{c?}
\]

\[
\Gamma \vdash ?F_{[w]}, \ \Delta - C \Downarrow L \quad \text{c?}
\]

where \( C' = \{ (a_1, a_2, \ldots a_n)_{[wx_1]} = (a_1, a_2, \ldots a_n)_{[wx_2]} \ \vee \\
(a_1, a_2, \ldots a_n)_{[wx_1]} = (0, 0, \ldots 0) \ \vee \\
(a_1, a_2, \ldots a_n)_{[wx_2]} = (0, 0, \ldots 0) \\
(a_1, a_2, \ldots a_n)_{[w]} = (a_1, a_2, \ldots a_n)_{[wx_1]} + (a_1, a_2, \ldots a_n)_{[wx_2]} \} \)
- Additive binary rules;

- Rules that ’distribute’ active formulae among the antecedent and succedent

- Rules with no additional constraints:
\[
\Gamma, \phi, \psi \vdash \Delta \quad - \mathcal{C} \upharpoonright \mathcal{L} \\
\Gamma, \phi \otimes \psi \vdash \Delta \quad - \mathcal{C} \upharpoonright \mathcal{L} \otimes L
\]
Example: labelled proof search

\[ \Pi_1 := \{ r \vdash r_1 \} \]

\[ \vdash \mathcal{L}_2 := \{ r \vdash r_1 \} \]

\[ \vdash w ? \mathcal{R} \]

\[ \vdash \mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \]

where \( \Pi_1 \) is

\[ \vdash - C_{\text{root}} \cup \{ t = 1, t_1 = 1, t \leq t_1 \} \]

\[ \vdash \mathcal{L}_1 := \{ t \vdash t_1 \} \]

\[ \text{Ax} \]

\[ C_{\text{root}} = \{ r + t + t_1 + q + p + s = 1 \} \]
Algorithm \( RE \) (input: labelled deduction \( \pi_l \))

1. Calculate possible assignments for the Boolean variables in \( \pi_l \);

2. If there is an assignment \( \mathcal{I} \) with at least one Boolean variable being assigned the value 0, then:
   
   2.1 Delete the atoms assigned 0 (i.e. delete the formulae made up of such atoms and the corresponding inferences).
   
   2.2 Delete all labels and constraints.

   EXIT: proof \( \pi' \).

   Else EXIT: ‘Simplification of proof \( \pi_l \) is not possible’
Final set of constraints: \( r + r_1 \leq q + p + s, \), 
\( s \leq q + p, \)  \( r = 1, \)  \( r_1 = 1, \)  \( t_1 = 1, \)  \( t = 1, \)  \( t \leq t_1 \)

1. Calculate possible assignments:

<table>
<thead>
<tr>
<th></th>
<th>p</th>
<th>q</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
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<tr>
<td>1</td>
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<td>1</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

2. Eliminate redundant formulae:

\[
\begin{align*}
&\frac{t \vdash t_1}{r,t \vdash t_1 \otimes q,r_1} \\
&\frac{r \vdash r_1}{r,t \vdash t_1 \otimes q,r_1} \quad \text{Ax} \\
&\frac{r \vdash q,r_1}{r,t \vdash t_1 \otimes q,r_1} \quad \otimes R \\
\end{align*}
\]

\[
\begin{align*}
&\frac{t \vdash t_1}{r,t \vdash t_1 \otimes (q \oplus s),r_1} \\
&\frac{r \vdash q\oplus s,r_1}{r,t \vdash t_1 \otimes (q \oplus s),r_1} \quad \otimes R \\
\end{align*}
\]
CONCLUSIONS and FUTURE WORK

✓ A technical, syntax approach;

✓ There was no a priori commitment to a particular search strategies;

✓ Sound and complete Algorithm RE;

✓ Contribution to a library of automated support tools for reasoning about sequent calculi proof search

- Analysis of failed proof attempts

\[
\begin{align*}
\Gamma, p_i \vdash p_j, \Delta & \quad - \mathcal{C} \cup \{p_j = 1, p_i = 1\} \Downarrow \mathcal{L} := \{p_i \vdash p_j\} \\
\mathcal{C} & \Downarrow \mathcal{L} \quad qAx
\end{align*}
\]

- Complexity analysis

- Combine Algorithm RE with our procedure for loop detection during proof search