Intelligent Geometrical Software

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Today's Agenda

- Predrag: Dynamic Geometrical Software and GCLC
- Predrag: Automated Theorem Proving in Geometry
- Pedro: GeoThms – A Repository of Geometrical Constructions
- Vesna & Sana: Formalization and Automation of Euclidean Geometry
- Predrag & Vesna & Sana: GCLC Lab Session

A Few Words about the ARGO Group

- The name and the game: Automated Reasoning GrOup
- Research interests: automated theorem proving and formal theorem proving, with emphasis on SAT and SMT solving, geometrical reasoning and their applications.
- URL: http://argo.matf.bg.ac.yu/
- Come to our seminar!

First Lecture:
Dynamic Geometrical Software and GCLC

Agenda

- What is dynamic geometry software?
- What are the DG tools?
- What is GCLC?
- Brief tutorial on GCLC

What is Dynamic Geometry Software?

- Interactive geometry software or Dynamic geometry software or Dynamic geometry environments or Dynamic geometry tools

- DG tools allow the user "to create and then manipulate geometric constructions, primarily in plane geometry".

- The user typically starts a construction with a few points, construct new objects, and then can move the points to see how the construction changes.

What Good is Dynamic Geometry Software?

- Fun and good for exploring geometry and mathematics

- Good for students:

  - to explore and understand the underlying principles of Euclidean constructions and transformations

  - to create and explore mathematical animations

What Good is Dynamic Geometry Software? (2)

- Good for teachers:
  - to demonstrate and illustrate concepts
  - to help students grasp the abstract concepts in mathematics

- Good for publishing:
  - easy producing complex mathematical figures

Some Commercial Dynamic Geometry Tools

- Cabri Geometry — since 1988

- Geometer Sketchpad (GSP) — since 1991

- Cinderella (different geometries)

Some Free Dynamic Geometry Tools

- KSEG

- Eukleides

- DrGeo

- http://en.wikipedia.org/wiki/Dynamic_geometry_software

Some of 3D Dynamic Geometry Tools

- Cabri 3D
- Archimedes Geo3D
- JavaView

Different Tools, Different Skills

- Animations, loci, ...
- Symbolic expressions, calculations, ...
- Saving constructions, saving figures, ...
- Multilingual
- Automated theorem proving, probabilistic proofs, ...

GCLC/WinGCLC

- First version released in 1996, originally, as a tool for producing geometrical illustrations for LaTeX, hence the name GCLC:

  "Geometry Constructions → LaTeX Converter".

- Command-line versions for Windows and Linux and a version with graphical interface for Windows (WinGCLC)

- Freely available from http://www.matf.bg.ac.yu/~janicic/gclc and from EMIS (The European Mathematical Information Service) servers http://www.emis.de/misc/index.html

The Main Purposes of GCLC/WinGCLC

- Visualizing geometry but also other fields of mathematics

- Use in mathematical education, in studying geometry and as a research tool

- Producing digital mathematical illustrations of high quality

GCLC Users

- Used in high-schools and university courses, and for publishing worldwide
- >18000 visitors since 2003, last 350 visitors (last three weeks):

GCLC: Basic Principles

- A construction is a formal procedure, not an image
- Producing mathematical illustrations should be based on "describing figures", not on "drawing figures" (similarly as TEX)
- Images can be produced from descriptions, but not vice-versa!
- All instructions are given explicitly, in GCLC language
- GCLC language is like a simple programming language, easily understandable to mathematicians

Features (part I)

- Support for geometrical constructions: sequences of primitive construction steps performed by ruler and compass
- Support for compound constructions and transformations
- Symbolic expressions, while-loops, user-defined procedures
- Conics, 2D and 3D curves, 3D surfaces
- Built-in theorem provers

Features (part II)

- User-friendly interface, interactive work, animations, traces
- Import from JavaView
- Export to different formats (LaTeX— several versions, EPS, BMP, SVG)
- Full XML support
- Free, small in size (750Kb–1100Kb), easy to use

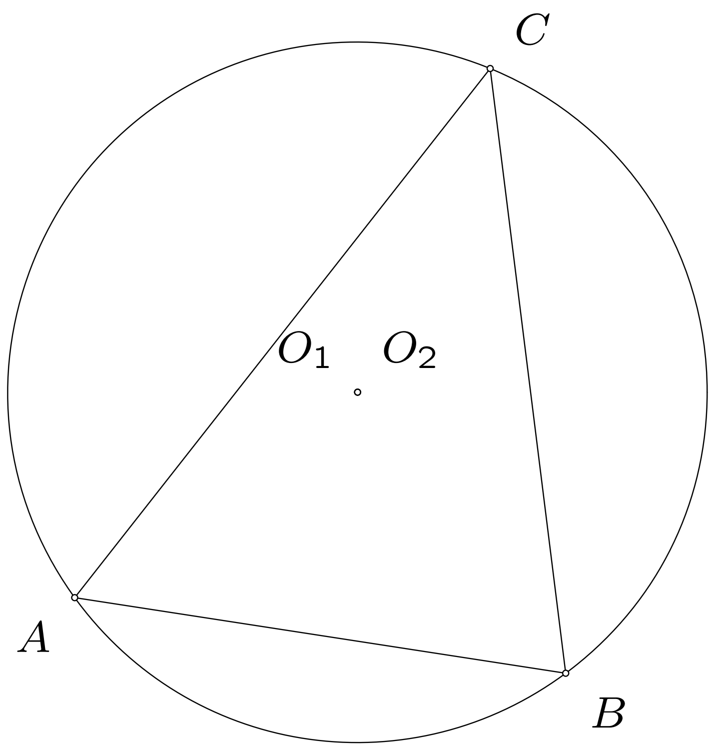
GCLC Language

- Instructions for describing **content**
- Instructions for describing **presentation**
- All of them are explicit, given within GCLC documents

Simple Example (part I)

```
% fixed points                       |     % labelling points
point A 15 20                        |     cmark_lb A
point B 80 10                        |     cmark_rb B
point C 70 90                        |     cmark_rt C
                                     |     cmark_lt O_1
% side bisectors                     |     cmark_rt O_2
med a B C                            |
med b A C                            |     % drawing the sides of the triangle ABC
med c B A                            |     drawsegment A B
                                     |     drawsegment A C
% intersections of bisectors         |     drawsegment B C
intersection O_1 a b                 |
intersection O_2 a c                 |     % drawing the circumcircle of the triangle
                                     |     drawcircle O_1 A
```

Simple Example (part II)



GCLC: A Brief Tutorial

- Basic definitions, constructions, transformations
- Drawing, labelling, and printing commands
- 2D and 3D Cartesian commands
- Symbolic expressions, loops, user-defined procedures
- Commands for describing animations

Second Lecture: Automated Theorem Proving in Geometry

Early History of Automated Theorem Proving in Geometry

- Euclid's *Elements*

- Hilber's *Foundations of Geometry*

- Tarski's elementary geometry

Geometrical Theorems of Constructive Type

- Conjectures that corresponds to properties of constructions

- Usually, only Euclidean plane geometry

- Non-degenerate conditions are very important

Coordinate-free methods

- Give traditional (human readable) proofs:
  - Gelertner's theorem prover (Gelertner 1950's)
  - Area method (Chou et.al.1992)
  - Angle method (Chou et.al.1990's)
  - ...

Coordinate-based methods

- Algebraic methods (no geometrical proofs, just algebraic arguments):
  - Gröbner basis method (Buchberger 1965)
  - Wu's method (Wu 1977)
  - ...

Area method

The method deals with the following geometry quantities:

ratio of directed segments: for four collinear points P, Q, A, and B such that $A \neq B$, it is the ratio $\frac{\overrightarrow{PQ}}{\overrightarrow{AB}}$;

signed area: it is the signed area S_{ABC} of a triangle ABC or the signed area S_{ABCD} of a quadrilateral $ABCD$;

Area method (2)

Pythagoras difference: for three points, P_{ABC} is defined as follows:

$$P_{ABC} = AB^2 + CB^2 - AC^2 \ .$$

Pythagoras difference for four points, P_{ABCD} is defined as follows:

$$P_{ABCD} = P_{ABD} - P_{CBD} \ .$$

real number: it is a real number, constant.

Area method (3)

- All construction steps are reduced to a limited number of specific constructions

- The conjecture is also expressed as an equality over geometry quantities (over points already introduced)

- The goal is to prove the conjecture by reducing it to a trivial equality (0=0)

Area method (4)

- For reducing the goal, different simplifications are used:

$$
\begin{array}{rcl}
x \cdot 1 & \rightarrow & x \\
x \cdot 0 & \rightarrow & 0 \\
S_{AAB} & \rightarrow & 0 \\
S_{ABC} & \rightarrow & S_{BCA}
\end{array}
$$

- Crucially, for each pair quantity-construction step there is one *elimination lemma* that enable eliminating a relevant point

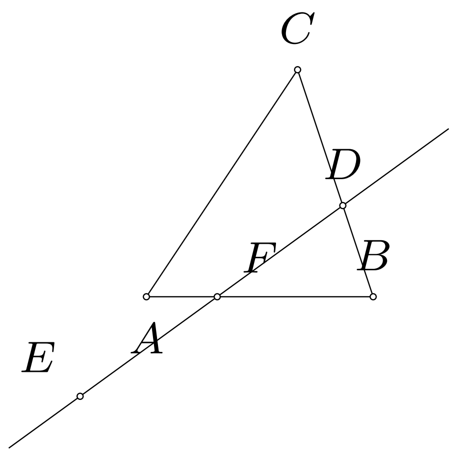
- Thank to these lemmas, the point are eliminated from the conjecture in opposite direction that they were introduced one by one

Area Method — Elimination lemmas

For instance, if a point Y is introduced as the intersection of lines UV and PQ, then Y can be eliminated from expression of the form $\frac{\overrightarrow{AY}}{\overrightarrow{CD}}$ using the following equality:

$$\frac{\overrightarrow{AY}}{\overrightarrow{CD}} = \begin{cases} \frac{S_{APQ}}{S_{CPDQ}}, & \text{if } A \in UV \\ \frac{S_{AUV}}{S_{CUDV}}, & \text{if } A \notin UV \end{cases}$$

Example: Menelaus's Theorem



- Conjecture:

$$\frac{\overrightarrow{AF}}{\overrightarrow{FB}} \cdot \frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}} = -1$$

Example: Menelaus's Theorem (2)

- Fragment of the proof:

  $\left(\frac{\overrightarrow{AF}}{\overrightarrow{BF}} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right)\right) = 1$, by algebraic simplifications

  $\left(\frac{S_{ADE}}{S_{BDE}} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right)\right) = 1$, by Lemma 8 (point F eliminated)

  ...

  $0 = 0$, by algebraic simplifications

Algebraic methods

- Geometry statements are of the equality form

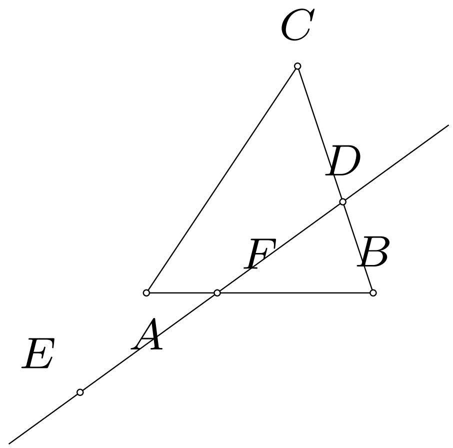
- Construction steps are converted into a polynomial system

$$
\begin{array}{rcl}
h_1(u_1, u_2, \ldots, u_d, x_1, \ldots, x_n) & = & 0 \\
h_2(u_1, u_2, \ldots, u_d, x_1, \ldots, x_n) & = & 0 \\
\ldots & & \\
h_t(u_1, u_2, \ldots, u_d, x_1, \ldots, x_n) & = & 0
\end{array}
$$

- The goal is to check whether for the conjecture it holds that

$$g(u_1, u_2, \ldots, u_d, x_1, \ldots, x_n) = 0$$

Example: Menelaus Theorem



- Coordinates assigned to the points:

$$A(0,0), B(u_1,0), C(u_2,u_3), D(x_1,u_4), E(x_2,u_5), F(x_4,0)$$

Example: Menelaus Theorem (2)

- Conditions:

  D on BC: $p_1 = -u_3x_1 + (u_4u_2 - u_4u_1 + u_3u_1)$

  E on AC: $p_2 = -u_3x_2 + u_5u_2$

  F on DE: $p_3 = (-u_5 + u_4)x_4 - u_4x_2 + u_5x_1$

- Conjecture:

$$p_4 = (-u_5u_3 + u_4u_3)x_4 + (-u_5u_4u_1 + u_5u_3u_1)$$

Example: Menelaus Theorem (3)

- After triangulation:

$$
\begin{aligned}
p_1 &= -u_3 x_1 + (u_4 u_2 - u_4 u_1 + u_3 u_1) \\
p_2 &= -u_3 x_2 + u_5 u_2 \\
p_3 &= (-u_5 + u_4) x_4 - u_4 x_2 + u_5 x_1
\end{aligned}
$$

- Wu's elimination procedure in several steps gives $p_4 = 0$, which was required to prove

Buchberger Method

- It builds a Gröbner bases (GB) for the set of polynomials corresponding to the construction

- Then it checks the conjecture, by efficiently testing whether its remainder with respect to GB is 0

Theorem Provers Built-into GCLC

- There are three theorem provers built-into GCLC:
  - a theorem prover based on the area method
  - a theorem prover based on the Wu's method
  - a theorem prover based on the Buchberger's method

- All of them are very efficient and can prove many non-trivial theorems in only seconds.

Using Theorem Provers Built-into GCLC

- The theorem provers are tightly built-in: the user has just to state the conjecture about the construction described.

- For example, in one of the above examples, points `O_1` and `O_2` are identical and this can be stated as follows

```
prove { identical O_1 O_2 }
```

Processing Descriptions of Constructions

- Syntactical check
- Semantical check (e.g., whether two concrete points determine a line)
- Deductive check — verifies if a construction is regular (e.g., whether two constructed points never determine a line)

Further Work

- Aiming at more intelligence in GCLC: e.g., solving constructive problems automatically

Conclusions

- Dynamic geometry tools are around for twenty years but just recently they started to be very intelligent

- Automated geometrical theorem provers are around for forty years but just recently they started to work in harmony with dynamic geometry tools

- GCLC aims to be a compact mathematical tool that combines ease of use and deep mathematical reasoning modules