Uniform Reduction to SAT and SMT

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Automated Reasoning GrOup (ARGO)

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Summer Research Institute (SuRI 2011)
School of Computer and Communication Sciences, EPFL,
Lausanne, Switzerland, June 20, 2011.
University of Belgrade
- Established in early 1800’s
- One of the oldest and largest in the region
- Around 90000 students and 4000 members of teaching staff

Faculty of Mathematics
- Around 1500 students and 80 members of teaching staff
- Departments for pure mathematics, computer science, astronomy...
Automated Reasoning GrOup (ARGO)

- Area: automated and interactive theorem proving, decision procedures, SAT, SMT, geometry reasoning
- 10 members
- COST Action IC0901 *Rich Model Toolkit* (chair Viktor Kuncak, EPFL)
- SCOPES Joint Research Project *Decision Procedures: from Formalizations to Applications* (with Viktor Kuncak and LARA group, EPFL)
- More at: [http://argo.matf.bg.ac.rs/](http://argo.matf.bg.ac.rs/)
Pseudorandom numbers can be generated using linear congruential generators:

\[ x_{n+1} \equiv ax_n + c \pmod{m} \]

where \( x_0 \) is the seed value (\( 0 \leq x_0 < m \)).

For example: \( x_{n+1} \equiv 1664525x_n + 1013904223 \pmod{2^{32}} \)

Given the seed, it is trivial to compute \( x_{100} \)

Given \( x_{100} \), how to compute the seed?
Problem SAT (SATisfiability)

Problem of deciding if a given propositional formula in CNF is satisfiable, i.e., if there is assignment to variables such that all clauses are true

Canonical NP-complete problem

Example: is \((p \lor q \lor \neg r) \land (p \lor \neg q \lor r) \land (p \lor \neg q \lor \neg r)\) satisfiable?

Can be reduced to any NP-complete problem and vice versa

Very efficient SAT solvers available (typically conflict-driven, clause-learning based)
Problem SMT (Satisfiability Modulo Theory)

- Problem of deciding if a given first-order formula is satisfiable with respect to combinations of background theories
- Examples of theories: linear arithmetic, uninterpreted functions, theories of data structures such as lists, arrays, bit vectors...
- Example: is $x \leq y \land y \leq x + c \land p(f(x) - f(y)) \land \neg p(0)$ satisfiable?
- Very efficient SMT solvers available (typically working in conjunction with SAT solvers)
SAT/SMT solvers are widely used, but encoding to SAT/SMT is typically made ad-hoc, by special-purpose tools.

There are interchange formats for SAT/SMT (e.g., SMT-lib) but no high-level specification languages.

No modelling and solving systems based on SMT.
Logical analysis of hash functions

- From early 2000’s, SAT was used in cryptanalysis
- Hash functions can be explored via SAT
- Rather explore hardness than solve the obtained instances
- Also: useful hard instances can be obtained
- Example problem: for given $y$, find $x$ such that $hash(x) = y$ (the hash function is preimage resistant if this is hard)
- How to encode problems as SAT instances?
for i from 0 to 79
    if 0 \leq i \leq 19 then
        f = (b and c) or (not b) and d
        k = 0x5A827999
    else if 20 \leq i \leq 39
        f = b xor c xor d
        k = 0x6ED9EBA1
    else if 40 \leq i \leq 59
        f = (b and c) or (b and d) or (c and d)
        k = 0x8F1BBCDC
    else if 60 \leq i \leq 79
        f = b xor c xor d
        k = 0xCA62C1D6
    temp = (a leftrotate 5) + f + e + k + w[i]
    e = d
    d = c
    c = b leftrotate 30
    b = a
    a = temp

Add this chunk's hash to result so far:
    h0 = h0 + a
    h1 = h1 + b
    h2 = h2 + c
    h3 = h3 + d
    h4 = h4 + e

Produce the final hash value (big-endian):
    digest = hash = h0 append h1 append h2 append h3 append h4
Obviously, analyzing the code and encoding in SAT by hand – tedious and error prone, instead:

- use the C-implementation
- represent variables (that are *unknowns*) by bitvectors (i.e., by vectors of propositional formulae)
- overload arithmetic operators and run the C++ code (as in symbolic execution)
- the given constraint evaluates to one propositional formula
- its model gives the values of the unknowns

Toy example

- **Alice picked a number and added 3. Then she doubled what she got. If the sum of the two numbers that Alice got is 12, what is the number that she picked?**

- The computation (if $A$ was given, it just tests if $A$ is indeed the required value)
  
  $B = A + 3$
  $C = 2 \times B$
  
  `assert(B+C==12);`

- The assertion evaluates to $A + 3 + 2 \times (A + 3) == 12$ and further to a SAT instance (if $A$ is represented as a bitvector)
Applicability of the idea

- Given implementation of $f : D \rightarrow D$, and given $y$, one can compute $x$ such that $f(x) = y$
- Example: the seed problem (the problem can be simply specified and solved, although not efficiently)
- Nice, but is this scope wide? Just for inverting functions?)
Applicability of the idea (2)

- **Version 1:** given \( f : D \rightarrow D \) and \( y \) one can compute \( x \) such that \( f(x) = y \)

- **Version 2:** given \( f : D \rightarrow \{0, 1\} \) one can compute \( x \), if it exists, such that \( f(x) = 1 \)

- **Version 3:** given \( f : D \rightarrow \{0, 1\} \) one can check if there is \( x \) such that \( f(x) = 1 \)

- Hence, suitable for solving NP-complete problems
Applicability of the idea (3)

- Given $f$ one can check if there is $x$ such that $f(x) = 1$ i.e., check if there are values that satisfy given conditions
- It is often easy to specify an (imperative) test if given values satisfy the conditions (i.e., to express $f$)
- It is often hard to develop an efficient specialized procedure that finds required values (i.e., to invert $f$)
Example

- Clique Problem: test whether a given graph contains a clique larger than a given size $k$
- **Hard**: check if there is such clique
- **Easy**: check if a given subgraph is a clique of the size $> k$
- A test if a given subgraph is a clique of the size $> k$ can serve as our specification
Example

- SAT Problem: test whether there is a valuation in which a given formula evaluates to true
- **Hard**: check if there is such valuation
- **Easy**: check if in a given valuation the formula evaluates to true
- A test if in a given valuation the formula evaluates to true can serve as our specification
Outline of the system

- Uniform Reduction to SAT
- Stand-alone system, implemented in C++
- C-like specification language
- Unknowns are represented as bit-vectors
- Specifications are symbolically executed
- Constraints give SAT instances
- A model of the SAT instance (if it exists) gives values of the unknowns
The C-like specification language supports:

- integer and Boolean data types; arrays
- implicit casting
- arithmetical, logical, relational and bit-wise operators
- flow-control statements (if, for, while) and functions

Restriction: conditions in for, while statements and array indices must not contain unknowns
Interpretation

- Specifications are symbolically executed
- The semantics is different from the standard semantics of imperative languages (e.g., undefined variables can be accessed)
- The result of the interpretation is a SAT instance
- If it is satisfiable, its models give solutions of the problem
The seed example

nX = nSeed;
for(nI = 0; nI < 100; nI++)
    nX = 1664525 * nX + 1013904223;
assert(nX==123);
CSP Example: The Eight Queens Puzzle

\begin{verbatim}
  nDim=8;
  bDomain = true;
  bNoCapture = true;
  for(ni=0; ni<nDim; ni++) {
    bDomain &&= (n[ni]<nDim);
    for(nj=0; nj<nDim; nj++)
      if(ni!=nj) {
        bNoCapture &&= (n[ni]!=n[nj]);
        bNoCapture &&= (ni+n[nj]!=nj+n[ni]) && (ni+n[ni] != nj+n[nj]);
      }
  }

  assert(bDomain && bNoCapture);
\end{verbatim}
Verification Example: Bit-counters

function nBC1(nX) {
    nBC1 = 0;
    for (nI = 0; nI < 16; nI++)
        nBC1 += nX & (1 << nI) ? 1 : 0;
}

function nBC2(nX) {
    nBC2 = nX;
    nBC2 = (nc2 & 0x5555) + (nc2>>1 & 0x5555);
    nBC2 = (nc2 & 0x3333) + (nc2>>2 & 0x3333);
    nBC2 = (nc2 & 0x0077) + (nc2>>4 & 0x0077);
    nBC2 = (nc2 & 0x000F) + (nc2>>8 & 0x000F);
}

assert(nBC1(nX)!=nBC2(nX));
 Beyond SAT: URSA Major — Overview of the system

- **URSA Major** (Uniform Reduction to SATisfiability Modulo Theory)
- The result of the interpretation is a SAT formula or a FOL formula in a SMT theory
- Basically the same principles as in URSA, but unknowns are not represented as vectors of propositional formula but as theory variables
- Currently several SAT solvers and several SMT solvers used
Overall Architecture

URSA MAJOR problem specification
↓ interpreter

Quantifier free FOL formula
↓ bitblasting

Propositional formula
↓ SAT solver ↓ SMT (BVA, LA, ...) solver

Values of unknowns/Solutions
Specifications may involve both interpreted (user-defined) and uninterpreted functions — thanks to the theory of uninterpreted function

Example:

\[
\text{assert}(x \neq y \ || \ f(x) == f(y));
\]

where \( f \) is not defined
Specifications may involve dealing with arrays, thanks to the theory of arrays

Example:
@nA = @nB;
@nB[3]=nX;
assert(@nA == @nB);
(@nA and @nB are arrays)
Comparison with related tools

- Comparable in efficiency to state-of-the-art constraint solvers (e.g., MiniZinc, OPL)
- Can express some constraints that other systems cannot (e.g., constraints over bitvectors, over arrays, involving modular arithmetic)
Comparison with related tools (2)

- One of distinguishing features:
  - The system is **declarative** (as no solving process has to be specified)
  - The system is **imperative** (as the specifications are given in the form of imperative tests)

- Can be viewed as a new programming paradigm
Conclusions

- A simple, high-level front-end to SAT/SMT solvers
- Suitable for solving a wide range of problems
- Different real-world applications
- Competitive to other modelling systems
- A novel (imperative-declarative) programming paradigm