Towards Understanding Triangle Construction Problems*

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Abstract. Straightedge and compass construction problems are one of the oldest and most challenging problems in elementary mathematics. The central challenge, for a human or for a computer program, in solving construction problems is a huge search space. In this paper we analyze one family of triangle construction problems, aiming at detecting a small core of the underlying geometry knowledge. The analysis leads to a small set of needed definitions, lemmas and primitive construction steps, and consequently, to a simple algorithm for automated solving of problems from this family. The same approach can be applied to other families of construction problems.

Keywords: Triangle construction problems, automated deduction in geometry

1 Introduction

Triangle construction problems (in Euclidean plane) are problems in which one has to construct, using straightedge[1] and compass,[2] a triangle that meets given (usually three) constraints [10, 24, 26]. The central problem, for a human or for a computer program, in solving triangle construction problems is a huge search space: primitive construction steps can be applied in a number of ways, exploding further along the construction. Consider, as an illustration, the following simple problem: *given the points A, B, and G, construct a triangle ABC such that G is its barycenter*. One possible solution is: construct the midpoint M_c of the segment AB and then construct a point C such that $\overrightarrow{M_cG}/\overrightarrow{M_cC} = 1/3$ (Figure 1). The solution is very simple and intuitive. However, if one wants to describe a systematic (e.g., automatic) way for reaching this solution, he/she should consider a wide range of possibilities. For instance, after constructing the point M_c,

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[1] The notion of "straightedge" is weaker than "ruler", as ruler is assumed to have markings which could be used to make measurements. Geometry constructions typically require use of straightedge and compass, not of ruler and compass.

[2] By compass, we mean *collapsible compass*. In contrast to *rigid compass*, one cannot use collapsible compass to "hold" the length while moving one point of the compass to another point. One can only use it to hold the radius while one point of the compass is fixed [3].

one might consider constructing midpoints of the segments AG, BG, or even of the segments AM_c, BM_c, GM_c, then midpoints of segments with endpoints among these points, etc. Instead of the step that introduces the point C such that $\overrightarrow{M_cG}/\overrightarrow{M_cC} = 1/3$ one may (unnecessarily) consider introducing a point X such that $\overrightarrow{AG}/\overrightarrow{AX} = 1/3$ or $\overrightarrow{BG}/\overrightarrow{BX} = 1/3$. Also, instead of the step that introduces the point C such that $\overrightarrow{M_cG}/\overrightarrow{M_cC} = 1/3$ one may consider introducing a point X such that $\overrightarrow{M_cG}/\overrightarrow{M_cX} = k$, where $k \neq 1/3$, etc. Therefore, this trivial example shows that any systematic solving of construction problems can face a huge search space even if only two high-level constructions steps that are really needed are considered. Additional problem in solving construction problems makes the fact that some of them are unsolvable (which can be proved by an algebraic argument), including, for instance, three antiquity geometric problems: circle squaring, cube duplication, angle trisection [31]. Although the problem of constructibility (using straightedge and compass) of a figure that can be specified by algebraic equations with coefficients in $\mathbf{Q}$ is decidable [14, 17, 23], there are no simple and efficient, "one-button" implemented decision procedures so, typically, proofs of insolvability of construction problems are made *ad-hoc* and not derived by uniform algorithms.

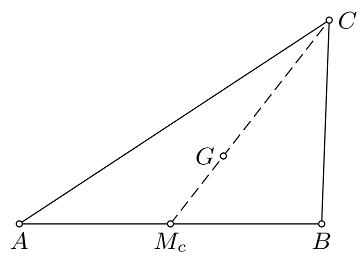

Fig. 1. Construction of a triangle ABC given its vertices A, B and the barycenter G

Construction problems have been studied, since ancient Greeks, for centuries and represent a challenging area even for experienced mathematicians. Since early twentieth century, "triangle geometry", including triangle construction problems, has not been considered a premier research field [8]. However, construction problems kept their role on all levels of mathematical education, almost in the same form for more than two and a half millenia, which make them probably the problems used most constantly throughout the history of mathematical education. Since the late twentieth century, geometry constructions are again a subject of research, but this time mainly meta-mathematical. There are two main lines of this work:

– **Research in axiomatic foundations of geometry constructions and foundational issues.** According to Pambuccian and his survey of axiomatizing geometric constructions, surprisingly, it is only in 1968 that geometric constructions became part of axiomatizations of geometry [28]. In constructive geometry axiomatizations, following ancient understanding, axioms are quantifier-free and involve only operation symbols (reflecting construction steps) and individual constants, in contrast to the Hilbert-style approach with relation symbols and where axioms are not universal statements. One such axiomatic theory of constructive geometry — **ECG** ("Elementary Constructive Geometry") was recently proposed by Beeson [3]. Constructive axiomatizations bring an alternative approach in geometry foundations, but they do not bring a substantial advantage to the Hilbert style when it comes to solving concrete construction problems.

– **Research in developing algorithms for solving construction problems.** There are several approaches for automated solving of construction problems [13, 15, 16, 29]. However, most, if not all of them, focus on search procedures and do not focus on finding a small portion of geometry knowledge (axioms and lemmas) that are underlying the constructions (although, naturally, all approaches have strict lists of available primitive construction steps). Earlier attempts at (manual) systematization of triangle construction problems also didn't provide small and clear, possibly minimal in some sense, lists of needed underlying geometry knowledge [11, 12, 24].

We find that it is important to locate, understand and systematize the knowledge relevant for solving construction problems or their subclasses. That should be useful for teachers, students and mathematical knowledge base generally. Also, such understanding should lead to a system that automatically solves these kinds of problems (and should be useful in education).

In this work we focus on one family of triangle construction problems and try to derive an algorithm for automated solving of problems based on a small portion of underlying geometry knowledge. Our analyses lead us to a small set of definitions, lemmas and construction rules needed for solving most of the solvable problems of this family. The same approach can be applied to other sorts of triangle construction problems and, more generally, to other sorts of construction problems. The approach, leading to a compact representation of the underlying geometry knowledge, can be seen not only as an algorithm for automated solving of triangle construction problems but also as a work in geometry knowledge management, providing a compact representation for a large sets of construction problems, currently not available in the literature.

2 Constructions by Straightedge and Compass

There are several closely related definitions of a notion of constructions by straightedge and compass [3, 9, 31]. By a straightedge-and-compass construction we will mean a sequence of the following primitive (or *elementary*) steps:

– construct an arbitrary point (possibly distinct from some given points);
– construct (with *ruler*) the line passing through two given distinct points;
– construct (with *compass*) the circle centered at some point passing through another point;
– construct an intersection (if it exists) of two circles, two lines, or a line and a circle.

In describing geometrical constructions, both primitive and compound constructions can be used. A straightedge-and-compass construction problem is a problem in which one has to provide a straightedge-and-compass construction such that the constructed objects meet given conditions. A solution of a geometrical construction problem traditionally includes the following four phases/components [1, 10, 18, 24]:

Analysis: In analysis one typically starts from the assumption that a certain geometrical object satisfies the specification Γ and proves that properties Λ enabling the construction also hold.

Construction: In this phase, straightedge-and-compass construction based on the analysis (i.e, on the properties Λ which are proved within it) has to be provided.

Proof: It this phase, it has to be proved that the provided straightedge-and-compass construction meets the given specification, i.e., the conditions Γ.

Discussion: In the discussion, it is considered how many possible solutions to the problem there exist and under which conditions.

3 Wernick's Problems

In 1982, Wernick presented a list of triangle construction problems [34]. In each problem, a task is to construct a triangle from three located points selected from the following set of 16 characteristic points:

- A, B, C, O: three vertices and circumcenter;
- M_a, M_b, M_c, G: the side midpoints and barycenter (centroid);
- H_a, H_b, H_c, H: three feet of altitudes and orthocenter;
- T_a, T_b, T_c, I: three feet of the internal angles bisectors, and incenter;

There are 560 triples of the above points, but Wernick's list consists only of 139 significantly different non-trivial problems. The triple (A, B, C) is trivial and, for instance, the problems (A, B, M_a), (A, B, M_b), (B, C, M_b), (B, C, M_c), (A, C, M_a), and (A, C, M_c) are considered to be symmetric (i.e., analogous). Some triples are redundant (e.g., (A, B, M_c) — given points A and B, the point M_c is uniquely determined, so it is redundant in (A, B, M_c)), but are still listed and marked **R** in Wernick's list. Some triples are constrained by specific conditions, for instance, in (A, B, O) the point O has to belong to the perpendicular bisector of AB (and in that case there are infinitely many solutions). In these problems, the locus restriction gives either infinitely many or no solutions. These problems are marked **L** in Wernick's list. There are 113 problems that do not belong to the groups marked **R** and **L**. Problems that can be solved by straightedge and ruler are marked **S** and problems that cannot be solved by straightedge and ruler are marked **U**.

In the original list, the problem 102 was erroneously marked **S** instead of **L** [27]. Wernick's list left 41 problem unresolved/unclassified, but the update from 1996 [27] left only 20 of them. In the meanwhile, the problems 90, 109,

1. A, B, O	L	36. A, M_b, T_c	S	71. O, G, H	R	106. M_a, H_b, T_c	U [27]
2. A, B, M_a	S	37. A, M_b, I	S	72. O, G, T_a	U [27]	107. M_a, H_b, I	U [27]
3. A, B, M_c	R	38. A, G, H_a	L	73. O, G, I	U [27]	108. M_a, H, T_a	U [27]
4. A, B, G	S	39. A, G, H_b	S	74. O, H_a, H_b	U [27]	109. M_a, H, T_b	U [30]
5. A, B, H_a	L	40. A, G, H	S	75. O, H_a, H	S	110. M_a, H, I	U [30]
6. A, B, H_c	L	41. A, G, T_a	S	76. O, H_a, T_a	S	111. M_a, T_a, T_b	U [30]
7. A, B, H	S	42. A, G, T_b	U [27]	77. O, H_a, T_b		112. M_a, T_a, I	S
8. A, B, T_a	S	43. A, G, I	S [27]	78. O, H_a, I		113. M_a, T_b, T_c	
9. A, B, T_c	L	44. A, H_a, H_b	S	79. O, H, T_a	U [27]	114. M_a, T_b, I	U [27]
10. A, B, I	S	45. A, H_a, H	L	80. O, H, I	U [27]	115. G, H_a, H_b	U [27]
11. A, O, M_a	S	46. A, H_a, T_a	L	81. O, T_a, T_b		116. G, H_a, H	S
12. A, O, M_b	L	47. A, H_a, T_b	S	82. O, T_a, I	S [27]	117. G, H_a, T_a	S
13. A, O, G	S	48. A, H_a, I	S	83. M_a, M_b, M_c	S	118. G, H_a, T_b	
14. A, O, H_a	S	49. A, H_b, H_c	S	84. M_a, M_b, G	S	119. G, H_a, I	
15. A, O, H_b	S	50. A, H_b, H	L	85. M_a, M_b, H_a	S	120. G, H, T_a	U [27]
16. A, O, H	S	51. A, H_b, T_a	S	86. M_a, M_b, H_c	S	121. G, H, I	U [27]
17. A, O, T_a	S	52. A, H_b, T_b	L	87. M_a, M_b, H	S [27]	122. G, T_a, T_b	
18. A, O, T_b	S	53. A, H_b, T_c	S	88. M_a, M_b, T_a	U [27]	123. G, T_a, I	
19. A, O, I	S	54. A, H_b, I	S	89. M_a, M_b, T_c	U [27]	124. H_a, H_b, H_c	S
20. A, M_a, M_b	S	55. A, H, T_a	S	90. M_a, M_b, I	U [30]	125. H_a, H_b, H	S
21. A, M_a, G	R	56. A, H, T_b	U [27]	91. M_a, G, H_a	L	126. H_a, H_b, T_a	S
22. A, M_a, H_a	L	57. A, H, I	S [27]	92. M_a, G, H_b	S	127. H_a, H_b, T_c	
23. A, M_a, H_b	S	58. A, T_a, T_b	S [27]	93. M_a, G, H	S	128. H_a, H_b, I	
24. A, M_a, H	S	59. A, T_a, I	L	94. M_a, G, T_a	S	129. H_a, H, T_a	L
25. A, M_a, T_a	S	60. A, T_b, T_c	S	95. M_a, G, T_b	U [27]	130. H_a, H, T_b	U [27]
26. A, M_a, T_b	U [27]	61. A, T_b, I	S	96. M_a, G, I	S [27]	131. H_a, H, I	S [27]
27. A, M_a, I	S [27]	62. O, M_a, M_b	S	97. M_a, H_a, H_b	S	132. H_a, T_a, T_b	
28. A, M_b, M_c	S	63. O, M_a, G	S	98. M_a, H_a, H	L	133. H_a, T_a, I	S
29. A, M_b, G	S	64. O, M_a, H_a	L	99. M_a, H_a, T_a	L	134. H_a, T_b, T_c	
30. A, M_b, H_a	L	65. O, M_a, H_b	S	100. M_a, H_a, T_b	U [27]	135. H_a, T_b, I	
31. A, M_b, H_b	L	66. O, M_a, H	S	101. M_a, H_a, I	S	136. H, T_a, T_b	
32. A, M_b, H_c	L	67. O, M_a, T_a	L	102. M_a, H_b, H_c	L	137. H, T_a, I	
33. A, M_b, H	S	68. O, M_a, T_b	U [27]	103. M_a, H_b, H	S	138. T_a, T_b, T_c	U [33]
34. A, M_b, T_a	S	69. O, M_a, I	S	104. M_a, H_b, T_a	S	139. T_a, T_b, I	S
35. A, M_b, T_b	L	70. O, G, H_a	S	105. M_a, H_b, T_b	S		

Table 1. Wernick's problems and their current status

110, 111 [30], and 138 [33] were proved to be unsolvable. Some of the problems were additionally considered for simpler solutions, like the problem 43 [2, 7], the problem 57 [35], or the problem 58 [6, 30]. Of course, many of the problems from Wernick's list were considered and solved along the centuries without the context of this list. The current status (to the best of our knowledge) of the problems from Wernick's list is given in Table 1: there are 72 **S** problems, 16 **U** problems, 3 **R** problems, and 23 **L** problems. Solutions for 59 solvable problems can be found on the Internet [30].

An extended list, involving four additional points (E_a, E_b, E_c — three Euler points, which are the midpoints between the vertices and the orthocenter and N — the center of the nine-point circle) was presented and partly solved by Connelly [5]. There are also other variations of Wernick's list, for instance, the list of problems to be solved given three out of the following 18 elements: sides a, b, c; angles α, β, γ; altitudes h_a, h_b, h_c; medians m_a, m_b, m_c; angle bisectors t_a, t_b, t_c; circumradius R; inradius r; and semiperimeter s. There are 186 significantly different non-trivial problems, and it was reported that (using Wernick's notation) 3 belong to the **R** group, 128 belong to the **S** group, 27 belong to the

U group, while the status of the remaining ones was unknown [4]. In addition to the above elements, radiuses of external incircles r_a, r_b, r_c and the triangle area S can be also considered, leading to the list of 22 elements and the total of 1540 triples. Lopes presented solutions to 371 non-symmetric problems of this type [24] and Fursenko considered the list of (both solvable and unsolvable) 350 problems of this type [11, 12].

4 Underlying Geometry Knowledge

Consider again the problem from Section 1 (it is problem 4 from Wernick's list). One solution is as follows: *construct the midpoint M_c of the segment AB and then construct a point C such that $\overrightarrow{M_cG}/\overrightarrow{M_cC} = 1/3$*. Notice that this solution implicitly or explicitly uses the following:

1. M_c is the side midpoint of AB (definition of M_c);
2. G is the barycenter of ABC (definition of G);
3. it holds that $\overrightarrow{M_cG}/\overrightarrow{M_cC} = 1/3$ (lemma);
4. it is possible to construct the midpoint of a line segment;
5. given points X and Y, it is possible to construct a point Z, such that $\overrightarrow{XY}/\overrightarrow{XZ} = 1/3$.

However, the nature of the above properties is typically not stressed within solutions of construction problems and the distinctions are assumed. Of course, given a proper proof that a construction meets the specification, this does not really affect the quality of the construction, but influences the meta-level understanding of the domain and solving techniques that it admits. Following our analyses of Wernick's list, we insist on a clear separation of concepts in the process of solving construction problems: separation into definitions, lemmas (geometry properties), and construction primitives. This separation will be also critical for automating the solving process.

Our analyses of available solutions of Wernick's problems[3] led to the list of 67 high-level construction rules, many of which were based on complex geometry properties and complex compound constructions. We implemented a simple forward chaining algorithms using these rules and it was able to solve each of solvable problems within 1s. Hence, the search over this list of rules is not problematic — what is problematic is how to represent the underlying geometry knowledge and derive this list. Hence, our next goal was to derive high-level construction steps from the (small) set of definitions, lemmas and construction primitives. For instance, from the following:

– it holds that $\overrightarrow{M_cG} = 1/3\overrightarrow{M_cC}$ (lemma);
– given points X, Y and U, and a rational number r, it is possible to construct a point Z such that: $\overrightarrow{XY}/\overrightarrow{UZ} = r$ (construction primitive; Figure 2);

[3] In addition to 59 solutions available from the Internet [30], we solved 6 problems, which leaves us with 7 solvable problems with no solutions.

we should be able to derive:

- given M_c and G, it is possible to construct C.

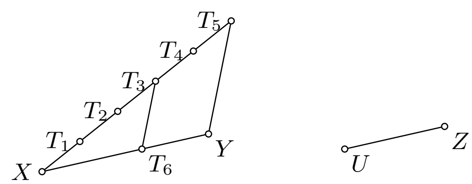

Fig. 2. Illustration for the construction: given points X, Y and U, and a rational number r, it is possible to construct a point Z such that: $\overrightarrow{XY}/\overrightarrow{UZ} = r$ (for $r = 5/3$)

After a careful study, we came to a relatively small list of geometry properties and primitive constructions needed. In the following, we list all definitions, geometry properties, and primitive constructions needed for solving most of the problems from Wernick's list that are currently known to be solvable. The following notation will be used: XY denotes the line passing through the distinct points X and Y; $\overline{XY}$ denotes the segment with endpoints X and Y; $\overrightarrow{XY}$ denotes the vector with endpoints X and Y; $k(X, Y)$ denotes the circle with center X that passes through Y; $\mathcal{H}(X, Y; Z, U)$ denotes that the pair of points X, Y is harmonically conjugate with the pair of points Z, U (i.e., $\overrightarrow{XU}/\overrightarrow{UY} = -\overrightarrow{XZ}/\overrightarrow{ZY}$); $s_p(X)$ denotes the image of X in line reflection with respect to a line p; $homothety_{X,r}(Y)$ denotes the image of Y in homothety with respect to a point X and a coefficient r.

4.1 Definitions

Before listing the definitions used, we stress that we find the standard definition of the barycenter (*the barycenter of a triangle is the intersection of the medians*) and the like — inadequate. Namely, this sort of definitions hides a non-trivial property that all three medians (the lines joining each vertex with the midpoint of the opposite side) do intersect in one point. Our, constructive version of the definition of the barycenter says that the barycenter G of a triangle ABC is the intersection of two medians AM_a and BM_b (it follows directly from Pasch's axiom that this intersection exists). Several of the definitions given below are formulated in this spirit. Notice that, following this approach, in contrast to the Wernick's criterion, for instance, the problems (A, B, G) and (A, C, G) are not symmetrical (but we do not revise Wernick's list).

For a triangle ABC we denote by (along Wernick's notation; Figure 3):

1. M_a, M_b, M_c (the side midpoints): points such that $\overrightarrow{BM_a}/\overrightarrow{BC} = 1/2$, $\overrightarrow{CM_b}/\overrightarrow{CA} = 1/2$, $\overrightarrow{AM_c}/\overrightarrow{AB} = 1/2$;
2. O (circumcenter): the intersection of lines perpendicular at M_a and M_b on BC and AC;
3. G (barycenter): the intersection of AM_a and BM_b;

4. H_a, H_b, H_c: intersections of the side perpendiculars with the opposite sides;
5. H (orthocenter): the intersection of AH_a and BH_b;
6. T_a, T_b, T_c: intersections of the internal angles bisectors with the opposite sides;
7. I (incenter): the intersection of AT_a and BT_b;
8. T'_a, T'_b, T'_c: intersections of the external angles bisectors with the opposite sides;
9. H'_{BC}, H'_{AC}, H'_{AB}: images of H in reflections over lines BC, AC and AB;
10. P_a, P_b, P_c: feet from I on BC, AC and AB;
11. N_a, N_b, N_c: intersections of OM_a and AT_a, OM_b and BT_b, OM_c and CT_c.

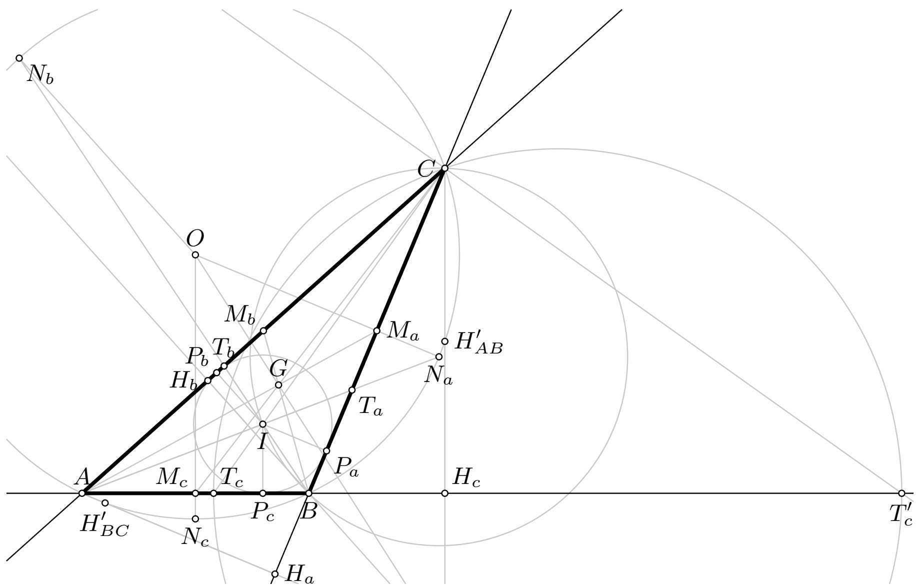

Fig. 3. Points used in solutions to Wernick's problems

4.2 Lemmas

For a triangle ABC it holds that (Figure 3):

1. O is on the line perpendicular at M_c on AB;
2. G is on CM_c;
3. H is on CH_c;
4. I is on CT_c;
5. B and C are on $k(O, A)$;
6. P_b and P_c are on $k(I, P_a)$;
7. $\overrightarrow{AG}/\overrightarrow{AM_a} = 2/3$, $\overrightarrow{BG}/\overrightarrow{BM_b} = 2/3$, $\overrightarrow{CG}/\overrightarrow{CM_c} = 2/3$;
8. $\overrightarrow{M_bM_a}/\overrightarrow{AB} = 1/2$, $\overrightarrow{M_cM_b}/\overrightarrow{BC} = 1/2$, $\overrightarrow{M_cM_a}/\overrightarrow{AC} = 1/2$;
9. $\overrightarrow{HG}/\overrightarrow{HO} = 2/3$;
10. $\overrightarrow{M_aO}/\overrightarrow{HA} = 1/2$, $\overrightarrow{M_bO}/\overrightarrow{HB} = 1/2$, $\overrightarrow{M_cO}/\overrightarrow{HC} = 1/2$;

11. AB, BC, CA touch $k(I, P_a)$;
12. N_a, N_b, N_c are on $k(O, A)$;
13. H'_{BC}, H'_{AC}, H'_{AB} are on $k(O, A)$;
14. C, H_b, H_c are on $k(M_a, B)$; A, H_a, H_c are on $k(M_b, C)$; B, H_a, H_b are on $k(M_c, A)$;
15. B, I are on $k(N_a, C)$; C, I are on $k(N_b, A)$; A, I are on $k(N_c, B)$;
16. AH, BH, CH are internal angles bisectors of the triangle $H_aH_bH_c$;
17. $\mathcal{H}(B, C; T_a, T'_a)$, $\mathcal{H}(A, C; T_b, T'_b)$, $\mathcal{H}(A, B; T_c, T'_c)$;
18. A is on the circle with diameter $T_aT'_a$; B is on the circle with diameter $T_bT'_b$; C is on the circle with diameter $T_cT'_c$;
19. $\angle T_cIT_b = \angle BAC/2+\pi/2$; $\angle T_bIT_a = \angle ACB/2+\pi/2$; $\angle T_aIT_c = \angle CBA/2+\pi/2$;
20. The center of a circle is on the side bisector of its arbitrary arc;
21. If the points X and Y are on a line p, so is their midpoint;
22. If $\overrightarrow{XY}/\overrightarrow{ZW} = r$ then $\overrightarrow{YX}/\overrightarrow{WZ} = r$;
23. If $\overrightarrow{XY}/\overrightarrow{ZW} = r$ then $\overrightarrow{ZW}/\overrightarrow{XY} = 1/r$;
24. If $\overrightarrow{XY}/\overrightarrow{ZW} = r$ then $\overrightarrow{WZ}/\overrightarrow{YX} = 1/r$;
25. If $\overrightarrow{XY}/\overrightarrow{XW} = r$ then $\overrightarrow{WY}/\overrightarrow{WX} = 1 - r$;
26. If $\mathcal{H}(X, Y; Z, W)$ then $\mathcal{H}(Y, X; W, Z)$;
27. If $\mathcal{H}(X, Y; Z, W)$ then $\mathcal{H}(Z, W; X, Y)$;
28. If $\mathcal{H}(X, Y; Z, W)$ then $\mathcal{H}(W, Z; Y, X)$;
29. If $\overrightarrow{XY}/\overrightarrow{XZ} = r$, Z is on p, and Y is not on p, then X is on $homothety_{Y,r/(1-r)}(p)$.

All listed lemmas are relatively simple and are often taught in primary or secondary schools within first lectures on "triangle geometry". They can be proved using a Hilbert's style geometry axioms or by algebraic theorem provers.

4.3 Primitive Constructions

We consider the following primitive construction steps:

1. Given distinct points X and Y it is possible to construct the line XY;
2. Given distinct points X and Y it is possible to construct $k(X, Y)$;
3. Given two distinct lines/a line and a circle/two distinct circles that intersect it is possible to construct their common point(s);
4. Given distinct points X and Y it is possible to construct the side bisector of $\overline{XY}$;
5. Given a point X and a line p it is possible to construct the line q that passes through X and is perpendicular to p;
6. Given distinct points X and Y it is possible to construct the circle with diameter $\overline{XY}$;
7. Given three distinct points it is possible to construct the circle that contains them all;
8. Given points X and Y and an angle α it is possible to construct the set of (all) points S such that $\angle XSY = \alpha$;
9. Given a point X and a line p it is possible to construct the point $s_p(X)$;

10. Given a line p and point X that does not lie on p it is possible to construct the circle k with center X that touches p;
11. Given a point X outside a circle k it is possible to construct the line p that passes through X and touches k;
12. Given two half-lines with the common initial point, it is possible to construct an angle congruent to the angle they constitute;
13. Given two intersecting lines it is possible to construct the bisector of internal angle they constitute;
14. Given one side of an angle and its internal angle bisector it is possible to construct the other side of the angle;
15. Given a point X, a line p and a rational number r, it is possible to construct the line $homothety_{X,r}(p)$;
16. Given points X, Y, and Z, and a rational number r it is possible to construct the point U such that $\overrightarrow{UX}/\overrightarrow{YZ} = r$.

All of the above construction steps can be (most of them trivially) expressed in terms of straightedge and compass operations. Still, for practical reasons, we use the above set instead of elementary straightedge and compass operations. These practical reasons are both more efficient search and simpler, high-level and more intuitive solutions.

5 Search Algorithm

Before the solving process, the preprocessing phase is performed on the above list of definitions and lemmas. For a fixed triangle ABC all points defined in Section 4.1 are uniquely determined (i.e., all definitions are instantiated). We distinguish between two types of lemmas:

instantiated lemmas: lemmas that describe properties of one or a couple of fixed objects (lemmas 1-20).

generic lemmas: lemmas given in an implication form (lemmas 21-29). These lemmas are given in a generic form and they are instantiated in a preprocessing phase by all objects satisfying their preconditions. So the instantiations are restricted with respect to the facts appearing in the definitions or lemmas.

Primitive constructions are given in a generic, non-instantiated form and they get instantiated while seeking for a construction sequence in the following manner: if there is an instantiation such that all objects from the premises of the primitive construction are already constructed (or given by a specification of the problem) then the instantiated object from the conclusion is constructed, if not already constructed. However, only a restricted set of objects is constructed – the objects appearing in some of the definitions or lemmas. For example, let us consider the primitive construction stating that for two given points it is possible to construct the bisector of the segment they constitute. If there would be no restrictions, the segment bisector would be constructed for each two constructed points, while many of them would not be used anywhere further. In contrast, this rule would be applied only to a segment for which its bisector occurs in some of the definitions or lemmas (for instance, when the endpoints of the segment belong to a circle, so the segment bisector gives a line to which the center of the circle belongs to). This can reduce search time significantly, as well as a length of generated constructions.

The goal of the search procedure is to reach all points required by the input problem (for instance, for all Wernick's problems, the goal is the same: construct a triangle ABC, i.e., the points A, B and C). The search procedure is iterative – in each step it tries to apply a primitive construction to the known objects (given by the problem specification or already constructed) and if it succeeds, the search restarts from the first primitive construction, in the waterfall manner. If all required points are constructed, the search stops. If no primitive construction can be applied, the procedure stops with a failure. The efficiency of solving, and also the found solution may depend on the order in which the primitive constructions are listed.

We implemented the above procedure in Prolog.[4] At this point the program can solve 58 Wernick's problems, each in less than 1s (for other solvable problems it needs some additional lemmas).[5] Of course, even with the above restricted search there are redundant construction steps performed during the search process and once the construction is completed, all these unnecessary steps are eliminated from the final, "clean" construction. The longest final construction consists of 11 primitive construction steps. Most of these "clean" constructions are the same as the ones that can be found in the literature. However, for problems with several different solutions, the one found by the system depends on the order of available primitive constructions/definitions/lemmas (one such example is given in Section 5.1). Along with the construction sequence, the set of non-degeneracy conditions (conditions that ensure that the constructed objects do exist, associated with some of construction primitives) is maintained.

5.1 Output

Once a required construction is found and simplified, it can be exported to different formats. Currently, export to simple natural language form is supported. For example, the construction for problem 7 (A, B, H) is represented as follows:

1. Using the point A and the point H, construct the line AH_a;
2. Using the point B and the point H, construct the line BH_b;
3. Using the point A and the line BH_b, construct the line AC;
4. Using the point B and the line AH_a, construct the line BC;
5. Using the line AC and the line BC, construct the point C.

[4] The source code is available at `http://argo.matf.bg.ac.rs/?content=downloads`.

[5] Currently, the program cannot solve the following solvable problems: 27, 43, 55, 57, 58, 69, 76, 82, 87, 96, 101, 126, 131, 139.

The generated construction can be (this is subject of our current work) also represented and illustrated (Figure 4) using the geometry language GCLC [21].

```
% free points
point A 5 5
point B 45 5
point H 32 18
% synthesized construction
line h_a A H
line h_b B H
perp b A h_b
perp a B h_a
intersec C a b
% drawing the triangle ABC
cmark_b A
cmark_b B
cmark_r C
cmark_b H
drawsegment A B
drawsegment A C
drawsegment B C
drawdashline h_a
drawdashline h_b
```

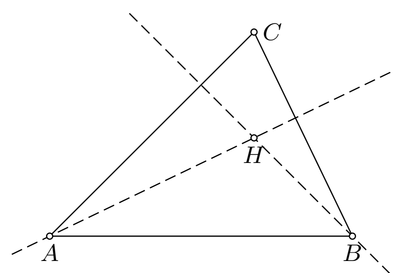

Fig. 4. A GCLC representation (left) and the corresponding illustration (right) for the solution to Wernick's problem 7

The above automatically generated solution is also example of a different (and simpler) solution from the one that can be found on the Internet [30] and that we used in building of our system (slightly reformulated in the way to use our set of primitive constructions):

1. Using the point A and the point B, construct the line AB;
2. Using the point H and the line AB, construct the line CH_c;
3. Using the line AB and the line CH_c, construct the point H_c;
4. Using the point H and the line AB, construct the point H'_{AB};
5. Using the point A, the point B and point H'_{AB}, construct the circle $k(O, A)$;
6. Using the circle $k(O, A)$ and the line CH_c, construct the point C.

5.2 Proving Constructions Correct

Generated constructions can be proved correct using provers available within the GCLC tool (the tool provide support for three methods for automated theorem proving in geometry: the area method, Wu's method, and the Gröbner bases method) [19, 20]. For instance, the construction given in Section 5.1 in GCLC terms, can be verified using the following additional GCLC code (note that the given coordinates of the points A, B and H are used only for generating an illustration and are not used in the proving process):

```
% definition of the orthocenter
line _a B C
perp _h_a A _a
line _b A C
perp _h_b B _b
intersec _H _h_a _h_b
% verification
prove { identical H _H }
% alternatively
% prove { perpendicular A H B C }
% prove { perpendicular B H A C }
```

The conjecture that H is indeed the orthocenter of ABC was proved by Wu's method in 0.01s. Instead of proving that H is identical to the orthocenter, one could prove that it meets all conditions from the definition of the orthocenter (which can be more suitable, in terms of efficiency, for automated theorem provers). For example, the area method proves such two conditions in 0.04s. It also returns non-degeneracy conditions [22] (needed in the discussion phase): A, B and H are not collinear, neither of the angles BAH, ABH is right angle (additional conditions, such as the condition that the lines a and b from the GCLC construction intersect, are consequences of these conditions). If A and B are distinct, and A and H are identical, then any point C on the line passing through A and perpendicular to AB makes a solution, and similar holds if B and H are identical. Otherwise, if A, B, and H are pairwise distinct and collinear or one of the angles BAH and ABH is right angle, there are no solutions.

5.3 Re-evaluation

The presented approach focuses on one sort of triangle construction problems, but it can be used for other sorts of problems. We re-evaluated our approach on another corpus of triangle construction problems (discussed in Section 3). In each problem from this corpus, a task is to construct a triangle given three lengths from the following set of 9 lengths of characteristic segments in the triangle:

1. $|AB|$, $|BC|$, $|AC|$: lengths of the sides;
2. $|AM_a|$, $|BM_b|$, $|CM_c|$: lengths of the medians;
3. $|AH_a|$, $|BH_b|$, $|CH_c|$: lengths of the altitudes.

There are 20 significantly different problems in this corpus and they are all solvable. This family of problems is substantially different from Wernick's problems: in Wernick's problems, the task is to construct a triangle based on the given, located points, while in these problems, the task is to construct a triangle with some quantities equal to the given ones (hence, the two solutions to the problem are considered identical if the obtained triangles are congruent). However, it turns out that the underlying geometry knowledge is mostly shared [4, 5, 11, 12, 24]. We succeeded to solve 17 problems from this family, using the system described above and additional 9 defined points, 2 lemmas, and 8 primitive constructions. Extensions of the above list of primitive constructions was expected because of introduction of segment measures. Since a search space was expanded by adding this new portion of knowledge, search times increased (the average solving time was 10s) and non-simplified constructions were typically longer than for the first corpus. However, simplified constructions are comparable in size with the ones from the first corpus and also readable.

6 Future Work

We plan to work on other corpora of triangle construction problems as well.[6] In order to control the search space, the solving system should first detect the family to which the problem belongs and use only the corresponding rules. Apart from detecting needed high-level lemmas and rules, we will try to more deeply explore these lemmas and rules and derive them from (suitable) axioms and from elementary straightedge and compass construction steps.

The presented system synthesizes a construction and can use an external automated theorem prover to prove that the construction meets the specification (as described in Section 5; full automation of linking the solver with automated theorem provers is subject of our current work). However, the provers prove only statements of the form: "if the conditions Γ are met, then the specification is met as well". They cannot, in a general case, check if the construction, the conditions Γ, are consistent (i.e., if the points that are constructed do exist). For instance, some provers cannot check if an intersection of two circles always exist. We are planning to use proof assistants and our automated theorem prover for coherent logic [32] for proving that the constructed points indeed exist (under generated non-degenerate conditions). With the verified theorem prover based on the area method [22] or with (more efficient) algebraic theorem proving verified by certificates [25], this would lead to completely machine verifiable solutions of triangle construction problems.

7 Conclusions

In our work we set up rather concrete tasks: (*i*) detect geometry knowledge needed for solving one of the most studied problems in mathematical education — triangle construction problems; (*ii*) develop a practical system for solving most of these problems. To our knowledge, this is the first systematic approach to deal with one family of problems (more focused than general construction problems) and to systemize underlying geometric knowledge. Our current results lead to a relatively small set of needed definitions, lemmas, and suitable primitive constructions and to a simple solving procedure. Generated constructions can be verified using external automated theorem provers. We believe that

[6] In our preliminary experiments, our system solved all triangle construction problems (5 out of 25) in the corpus considered by Gulwani et.al [16]; our system can currently solve only a fraction of 135 problems considered by Gao and Chou [13], since most of them are not triangle construction problems or involve the knowledge still not supported by our system.

any practical solver would need to treat this detected geometry knowledge one way or another (but trading off with efficiency). We expect that limited additions to the the geometry knowledge presented here would enable solving most of triangle construction problems appearing in the literature.

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