# Constructibility Classes for Triangle Location Problems

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**Abstract.** Straightedge-and-compass construction problems are well known for different reasons. One of them is the difficulty to prove that a problem is not constructible: it took about two millennia to prove that it is not possible in general to cut an angle into three equal parts by using only straightedge and compass. Today, such proofs rely on algebraic tools difficult to apprehend by high school student. On the other hand, the technique of problem reduction is often used in theory of computation to prove other kinds of impossibility. In this paper, we adapt the notion of reduction to geometric constructions in order to have geometric proofs for unconstructibility based on a set of problems known to be unconstructible. Geometric reductions can also be used with constructible problems: in this case, besides having constructibility, the reduction also yields a construction. To make the things concrete, we focus this study to a corpus of triangle location problems proposed by William Wernick in the eighties.

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## 1. Introduction

A straightedge-and-compass construction problem is a problem of finding a construction in Euclidean plane using only straightedge and compass such that the constructed objects meet given conditions [25, 20, 9]. In such constructions, only some elementary operations are allowed: draw a line through two points, draw a circle knowing its center and passing through a point, construct the intersection of already constructed lines and circles (a more precise definition of straightedge-and-compass construction problems is given in Section 1.1). Here is a classical example of such problem:

*Example* 1. Given two distinct points $A$ and $B$ and a circle $k$, construct using only straightedge and compass a line $d$ passing through $A$ and cutting $k$ in two points $E$ and $F$ such that $EB = FB$.

In the following text, instead of *a straightedge-and-compass construction* we will write shortly *a construction*, instead of *straightedge-and-compass constructibility* we will write shortly *constructibility*, etc.

Construction problems have been studied for thousand of years and they have been an endless source of inspiration for high school exercises in geometry. The central challenge, for a human or for a computer program, in solving construction problems is a huge search space: elementary

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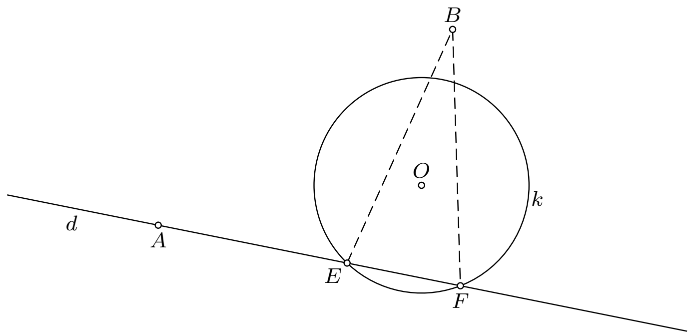


FIGURE 1. Illustration for Example 1.

construction steps can be applied in a number of ways, exploding further along the construction. There is a number of approaches for solving construction problems, including approaches that have been automated. Some of these approaches are geometry-based [3, 17, 23, 27, 29, 16, 12, 28], and some are algebra-based [4, 13].

Straightedge-and-compass constructions are also well-known in epistemology of mathematics, for instance by famous unconstructible problems, like the circle squaring problem, which was open problem for almost two millennia.

Within this paper we present a complete study of a concrete domain of construction problems, with proofs that are readable by high-school students by assuming only few results of unconstructibility. We focus on a corpus of problems proposed by William Wernick, which consists of a set of 139 triangle location problems [34]. Each problem from Wernick's list requires constructing a triangle $ABC$ given three characteristic points like, for instance, centroid, side midpoints, and orthocenter (Table 1). For this kind of problems, we define an adapted notion of *reduction* (by straightedge and compass) and we use it to compute a *complete set* of problems whose status is known and such that the status of every problem in Wernick's list can be found by reduction (without using advanced algebraic computations).

This paper is organized as follows. The rest of this section is devoted to recall some basics for straightedge and compass constructions and it introduces the notion of reduction. Section 2 describes a set of triangle location problems proposed by Wernick. Section 3 introduces the notion of reduction and of complete set within a geometric construction problems corpus. In section 4, we show how these concepts apply to Wernick's list by computing a complete set for it. Section 5 concludes by giving some tracks for generalizing these results.

### 1.1. Straightedge-and-compass Constructions

There are several closely related definitions of a notion of constructions by straightedge and compass [8, 32, 2]. The following one is a simple definition which can be found in standard textbooks.

**Definition 1.** An elementary construction step (with straightedge and compass) is one of the following constructions:

- construct an arbitrary point (possibly distinct from some given points)[1];
- construct (by the *straightedge*) a line passing through two given distinct points;
- construct (by the *compass*) a circle with its centre at some point and passing through another point;

[1]A construction of an arbitrary point on a given line/circle can also be used. In our work presented in this article, we used it within solutions of locus dependent problems.

- construct the intersection (if it exists) of two circles, two lines, or of a line and a circle.

**Definition 2.** Given a finite set of points $S$, a point $P$ is *constructible from the set* $S$ if there is a finite set of points $\{P_0, \dots, P_n\}$ such that $P = P_n$ and each point $P_i$ ($0 \leq i \leq n$) is either a point from $S$ or, when $i > 0$, is obtained by an elementary construction step from $\{P_0, \dots, P_{i-1}\}$.

A construction problem consists of the specification of a figure, a statement in mathematical terms, where some objects are given and others are to be constructed in the sense of the previous definition. This kind of problems is closely related to geometric constraint satisfaction problem [19] where the specification of a figure is formalized by the notion of *system of constraints*, which is a triple $\langle C, A, X \rangle$ where $C$ is a set of geometric constraints, $A$ is a set of parameters, that is – the given objects, and $X$ a set of unknowns, that is – the objects to be found. The solutions of a geometric constraint system are sought in Euclidean plane, a priori without considering straightedge and compass constructibility. When there are no solutions, the system is said to be *over-constrained*; when there are infinitely many solutions the system is said to be *under-constrained*, and, otherwise, the system is said to be *well-constrained*.

It is important to understand that a problem can be well-constrained and not constructible: the solutions exist, but they cannot be constructed. On the opposite, a problem is constructible if it is well-constrained and the solutions are constructible. Things are a bit more complex due to the presence of parameters and the existence of degenerate cases. Let us consider the problem given in Example 1 (Figure 1). It is not difficult to see that the line $d$ has to be perpendicular to the line $BO$, where $O$ is the center of circle $k$ and its construction is straightforward. But, the points $E$ and $F$ exist only when $A$ is between the two lines tangent to $k$ and perpendicular to $BO$: this system is well-constrained if $A$ is between these lines and over-constrained otherwise. There is also a special case when the points $B$ and $O$ are equal: the problem is then under-constrained. We say that a problem is generically well-constrained if there is an open set, in the metric space of parameters, where the problem is well-constrained [15]. Then, a problem is constructible if it is generically well-constrained and when solutions exist, they are constructible.

### 1.2. Reduction

For pedagogical reasons, the construction problems encountered in education are more often constructible than unconstructible: indeed, it is easier to prove constructibility by exhibiting a construction than to prove that there is no such construction. Usually, proofs of unconstructibility rely on algebraic notions that are beyond the usual high-school mathematical background. One of our goals is producing a complete study of a concrete domain of constructions, with proofs that are readable by high-school students by assuming only few results of unconstructibility. In this matter, our main idea comes from the proofs of undecidability in the computability theory, which is another domain where it is difficult to prove that there is no program able to achieve a given task. Indeed, in that field, proving that a problem is undecidable passes almost always through *reduction* of the given problem to another one which is known to be undecidable. The very root of this "tree of reductions" is the famous halting problem. Notice that there are also specialized reductions in the complexity theory and the core sets of problems (such as NP-complete problems), where some additional conditions are considered for the notion of reduction.

Both constructibility and unconstructibility can sometimes be proved using *reduction*, i.e., reducing the current problem to some other problem with a known status. This is illustrated by the following two examples (the latter one based on Archimedes' construction).

*Example* 2. One of the most classical examples of reduction of construction problems is perhaps the construction of a regular polygon with $2n$ sides which is easily reduced to the construction of a regular polygon with $n$ sides. As a consequence, a regular polygon with 10 sides is constructible (Figure 2 left).

*Example* 3. Given three non-collinear points $A$, $B$, and $O$, construct points $X$ and $Y$ such that (Figure 2, right):

- $OX \cong OB$,
- $XY \cong OB$,
- points $B$, $X$, and $Y$ are collinear, and
- points $A$, $O$, and $Y$ are collinear.

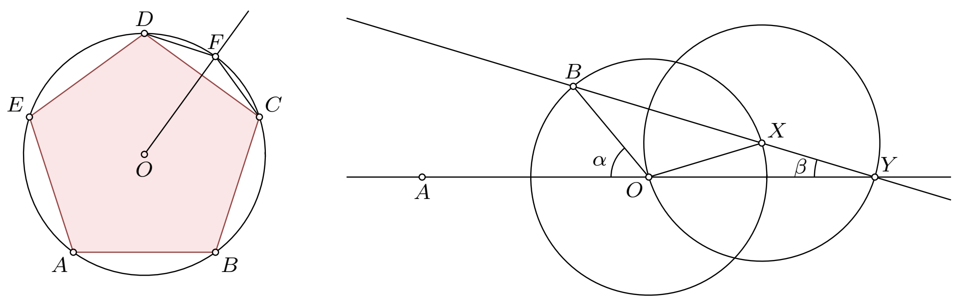


FIGURE 2. Examples of reductions: toward a construction (left), for proving unconstructibility (right).

Using elementary geometry, it can be proved that the angle $\alpha = \angle AOB$ is three times greater than the angle $\beta = \angle AYB$. Thus, if this problem is constructible, so is the angle trisection problem. However, it is well-known that, in general, one cannot divide an angle in three equal parts using only straightedge and compass.

As far as we know, none of computer-supported approaches for solving construction problems addresses the issue of reducibility between construction problems. In this paper, we will focus on solving construction problems by reduction and on defining certain classes of construction problems.

## 2. Wernick's List and Triangle Location Problems

In the folklore of geometric constructions, triangle location problems are very classical. Informally, the problem consists in constructing a triangle $ABC$ given three *characteristic points* like the vertices themselves, incenter, circumcenter, orthocenter, side midpoints, etc.

### 2.1. Wernick's List

One corpus of triangle location problems was presented by William Wernick in 1982. [34] (many of these problems were considered along the centuries, before this list was compiled). In each problem, the task is to construct a triangle $ABC$ starting from three located points selected from the following set of 16 characteristic points:

- $A$, $B$, $C$, $O$: three vertices and the circumcenter;
- $M_a$, $M_b$, $M_c$, $G$: the side midpoints and the centroid;
- $H_a$, $H_b$, $H_c$, $H$: three feet of altitudes and the orthocenter;
- $T_a$, $T_b$, $T_c$, $I$: three feet of the internal angles bisectors, and the incenter.

There are 560 triples of the above points, but Wernick's list consists only of 139 significantly different non-trivial problems. The triple $(A, B, C)$[2] is trivial and, for instance, the problems $(A, B, M_a)$, $(A, B, M_b)$, $(B, C, M_b)$, $(B, C, M_c)$, $(A, C, M_a)$, and $(A, C, M_c)$ are considered to be

[2]For denoting triples of Wernick's points, traditionally parentheses are used, although curly brackets would be more accurate (to stress that these points are elements of a set).

symmetric (i.e., analogous). There are variations of Wernick's list — involving additional points [5] or various geometrical quantities [20, 10, 11].

We adopt here the following, more formal, definitions and conventions for location triangle problems.

**Definition 3.** Let $\mathbb{E}$ be Euclidean plane, a function $t : \mathbb{E}^3 \to \mathbb{E}$ is a *triangular function* if it is invariant under the action of group of similarity transformations.

Given a triangle $ABC$ and a triangular function $t$, a point $t(A, B, C)$ is called a *characteristic point* of the triangle $ABC$.

By convention, the vertices $A$, $B$, and $C$ are considered characteristic points by the functions $V_a : (A, B, C) \mapsto A$, $V_b : (A, B, C) \mapsto B$ and $V_c : (A, B, C) \mapsto C$. When there is no ambiguity, we use the following naming convention: if $P$ is a characteristic point of $ABC$, its triangular function is named by corresponding lower case letter $p$ and we have the equality $P = p(A, B, C)$. For instance, if $M_a$ denotes the midpoint of the side $BC$, we will write $M_a = m_a(A, B, C)$[3].

**Definition 4.** A *triangle location problem* is a problem specified by three characteristic points $(P_1, P_2, P_3)$ and corresponds to the straightedge and compass resolvability of the constraint system $\langle\{P_1 = p_1(A, B, C), P_2 = p_2(A, B, C), P_3 = p_3(A, B, C)\}, (P_1, P_2, P_3), (A, B, C)\rangle$ for which only solutions where $A$, $B$, and $C$ are not collinear are required.

For example, $(M_a, I, O)$ is the problem of constructing a triangle $ABC$ given three points such that the first one is the midpoint of $BC$, the second and the third ones are respectively the incenter and the circumcenter of the triangle $ABC$.

### 2.2. Status of Triangle Location Problems

In Section 1.1, we introduced the notions of constructibility, in this section we will be more precise in the domain of triangle location problems. Since we consider problems with parameters, all constrainedness notions used below are the generic ones.

Each triangle location problem has many instances corresponding to positions of the given points. A triple of Wernick's points makes a Wernick's problem while each Wernick's problem has many instances – for all possible choices of the given points in Euclidean plane. As mentioned in the introduction, the notion well-constrainedness for these problem is generic. Then, according to the possibility of constructing all solutions using only straightedge and compass, Wernick used four statuses for the problems in his list. Here we introduce them in a more precise manner. For a given problem $P$:

- if *almost all*[4] its instances are well-constrained then
  - if whenever there is a required triangle $ABC$ – then it is straightedge-and-compass constructible,
    - we say that $P$ is **S** (solvable)
  - otherwise
    - we say that $P$ is **U** (unsolvable)
- otherwise
  - if whenever there is a required triangle $ABC$ – then one of the given points is uniquely determined by the other two,
    - we say that $P$ is **R** (redundant)
  - otherwise
    - we say that $P$ is **L** (locus dependent)

[3] Even if point $A$ is not involved in this definition, we put it in the settings for the sake of uniformity

[4] By "almost all instances of a triple of points meet a condition $X$", we mean: "if two given points are fixed in Euclidean plane, then the sets of positions of the third point such that $X$ does not hold has a measure zero in Euclidean plane".

Along with the list of problems, Wernick provided statuses for most of them [34]. In Wernick's list, the problem 102 was erroneously marked **S** instead of **L** [26], and the problem 108 was erroneously marked **U** instead of **S** [30]. Wernick's list left 41 problem unresolved/unclassified, but the update from 1996 [26] left only 20 of them. In the meanwhile, the problems 90, 109, 110, 111 [31], and 138 [33] were proved to be unsolvable. Some of the problems were additionally considered for simpler solutions, like the problem 43 [1, 7], the problem 57 [35], or the problem 58 [6, 31]. Finally, in [30], the status of all the problems has been set (Table 1): there are 74 **S** problems, 39 **U** problems, 3 **R** problems, and 23 **L** problems. Solutions for 59 solvable problems can be found on the Internet [31, 21].

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1. | $A, B, O$ | L | | 36. | $A, M_b, T_c$ | S | | 71. | $O, G, H$ | R | | 106. | $M_a, H_b, T_c$ | U | [26] |
| 2. | $A, B, M_a$ | S | | 37. | $A, M_b, I$ | S | | 72. | $O, G, T_a$ | U | [26] | 107. | $M_a, H_b, I$ | U | [26] |
| 3. | $A, B, M_c$ | R | | 38. | $A, G, H_a$ | L | | 73. | $O, G, I$ | U | [26] | 108. | $M_a, H, T_a$ | **S** | [30] |
| 4. | $A, B, G$ | S | | 39. | $A, G, H_b$ | S | | 74. | $O, H_a, H_b$ | U | [26] | 109. | $M_a, H, T_b$ | U | [31] |
| 5. | $A, B, H_a$ | L | | 40. | $A, G, H$ | S | | 75. | $O, H_a, H$ | S | | 110. | $M_a, H, I$ | U | [31] |
| 6. | $A, B, H_c$ | L | | 41. | $A, G, T_a$ | S | | 76. | $O, H_a, T_a$ | S | | 111. | $M_a, T_a, T_b$ | U | [31] |
| 7. | $A, B, H$ | S | | 42. | $A, G, T_b$ | U | [26] | 77. | $O, H_a, T_b$ | U | [30] | 112. | $M_a, T_a, I$ | S | |
| 8. | $A, B, T_a$ | S | | 43. | $A, G, I$ | S | [26] | 78. | $O, H_a, I$ | U | [30] | 113. | $M_a, T_b, T_c$ | U | [30] |
| 9. | $A, B, T_c$ | L | | 44. | $A, H_a, H_b$ | S | | 79. | $O, H, T_a$ | U | [26] | 114. | $M_a, T_b, I$ | U | [26] |
| 10. | $A, B, I$ | S | | 45. | $A, H_a, H$ | L | | 80. | $O, H, I$ | U | [26] | 115. | $G, H_a, H_b$ | U | [26] |
| 11. | $A, O, M_a$ | S | | 46. | $A, H_a, T_a$ | L | | 81. | $O, T_a, T_b$ | U | [30] | 116. | $G, H_a, H$ | S | |
| 12. | $A, O, M_b$ | L | | 47. | $A, H_a, T_b$ | S | | 82. | $O, T_a, I$ | S | [26] | 117. | $G, H_a, T_a$ | S | |
| 13. | $A, O, G$ | S | | 48. | $A, H_a, I$ | S | | 83. | $M_a, M_b, M_c$ | S | | 118. | $G, H_a, T_b$ | U | [30] |
| 14. | $A, O, H_a$ | S | | 49. | $A, H_b, H_c$ | S | | 84. | $M_a, M_b, G$ | S | | 119. | $G, H_a, I$ | S | [30] |
| 15. | $A, O, H_b$ | S | | 50. | $A, H_b, H$ | L | | 85. | $M_a, M_b, H_a$ | S | | 120. | $G, H, T_a$ | U | [26] |
| 16. | $A, O, H$ | S | | 51. | $A, H_b, T_a$ | S | | 86. | $M_a, M_b, H_c$ | S | | 121. | $G, H, I$ | U | [26] |
| 17. | $A, O, T_a$ | S | | 52. | $A, H_b, T_b$ | L | | 87. | $M_a, M_b, H$ | S | [26] | 122. | $G, T_a, T_b$ | U | [30] |
| 18. | $A, O, T_b$ | S | | 53. | $A, H_b, T_c$ | S | | 88. | $M_a, M_b, T_a$ | U | [26] | 123. | $G, T_a, I$ | U | [30] |
| 19. | $A, O, I$ | S | | 54. | $A, H_b, I$ | S | | 89. | $M_a, M_b, T_c$ | U | [26] | 124. | $H_a, H_b, H_c$ | S | |
| 20. | $A, M_a, M_b$ | S | | 55. | $A, H, T_a$ | S | | 90. | $M_a, M_b, I$ | U | [31] | 125. | $H_a, H_b, H$ | S | |
| 21. | $A, M_a, G$ | R | | 56. | $A, H, T_b$ | U | [26] | 91. | $M_a, G, H_a$ | L | | 126. | $H_a, H_b, T_a$ | S | |
| 22. | $A, M_a, H_a$ | L | | 57. | $A, H, I$ | S | [26] | 92. | $M_a, G, H_b$ | S | | 127. | $H_a, H_b, T_c$ | U | [30] |
| 23. | $A, M_a, H_b$ | S | | 58. | $A, T_a, T_b$ | S | [26] | 93. | $M_a, G, H$ | S | | 128. | $H_a, H_b, I$ | U | [30] |
| 24. | $A, M_a, H$ | S | | 59. | $A, T_a, I$ | L | | 94. | $M_a, G, T_a$ | S | | 129. | $H_a, H, T_a$ | L | |
| 25. | $A, M_a, T_a$ | S | | 60. | $A, T_b, T_c$ | S | | 95. | $M_a, G, T_b$ | U | [26] | 130. | $H_a, H, T_b$ | U | [26] |
| 26. | $A, M_a, T_b$ | U | [26] | 61. | $A, T_b, I$ | S | | 96. | $M_a, G, I$ | S | [26] | 131. | $H_a, H, I$ | S | [26] |
| 27. | $A, M_a, I$ | S | [26] | 62. | $O, M_a, M_b$ | S | | 97. | $M_a, H_a, H_b$ | S | | 132. | $H_a, T_a, T_b$ | U | [30] |
| 28. | $A, M_b, M_c$ | S | | 63. | $O, M_a, G$ | S | | 98. | $M_a, H_a, H$ | L | | 133. | $H_a, T_a, I$ | S | |
| 29. | $A, M_b, G$ | S | | 64. | $O, M_a, H_a$ | L | | 99. | $M_a, H_a, T_a$ | L | | 134. | $H_a, T_b, T_c$ | U | [30] |
| 30. | $A, M_b, H_a$ | L | | 65. | $O, M_a, H_b$ | S | | 100. | $M_a, H_a, T_b$ | U | [26] | 135. | $H_a, T_b, I$ | U | [30] |
| 31. | $A, M_b, H_b$ | L | | 66. | $O, M_a, H$ | S | | 101. | $M_a, H_a, I$ | S | | 136. | $H, T_a, T_b$ | U | [30] |
| 32. | $A, M_b, H_c$ | L | | 67. | $O, M_a, T_a$ | L | | 102. | $M_a, H_b, H_c$ | L | | 137. | $H, T_a, I$ | U | [30] |
| 33. | $A, M_b, H$ | S | | 68. | $O, M_a, T_b$ | U | [26] | 103. | $M_a, H_b, H$ | S | | 138. | $T_a, T_b, T_c$ | U | [33] |
| 34. | $A, M_b, T_a$ | S | | 69. | $O, M_a, I$ | S | | 104. | $M_a, H_b, T_a$ | S | | 139. | $T_a, T_b, I$ | S | |
| 35. | $A, M_b, T_b$ | L | | 70. | $O, G, H_a$ | S | | 105. | $M_a, H_b, T_b$ | S | | | | | |

TABLE 1. The Wernick's list with the status for all the problems.

## 3. Reduction Relations

A reduction is a way of transforming a problem $P$ into a problem $P'$ such that certain properties are preserved. In our case, the properties consist in characteristic points and straightedge and compass resolvability. We summarize this in the definition:

**Definition 5 (Relation $\mapsto$).** Given triangle location problems $P : (P_1, P_2, P_3)$ and $P' : (P'_1, P'_2, P'_3)$, we say that there is a straightedge and compass reduction from $P$ to $P'$, or more simply that $P$ *reduces to* $P'$ and we write $P \mapsto P'$, if each of the points $P'$ can be constructed from the points $P$.

The problem of constructing $P'$ from $P$ corresponds to the following constraint system

$$\langle \left\{ \begin{array}{l} P_1 = p_1(A,B,C) \\ P_2 = p_2(A,B,C) \\ P_3 = p_3(A,B,C) \\ P'_1 = p'_1(A,B,C) \\ P'_2 = p'_2(A,B,C) \\ P'_3 = p'_3(A,B,C) \end{array} \right. , (P_1,P_2,P_3), (P'_1,P'_2,P'_3) \rangle$$

where $A, B$, and $C$ are variables which have to be eliminated either in an algebraic sense or by using some geometric knowledge. It can also be expressed in logical terms in the style we used for expressing correctness of constructions [24].

Straightforwardly, since a construction gives the solutions to a constraint system, if $P \mapsto P'$, and if $P'$ has solutions then $P$ also has solutions. It is easy to see in this context that if $P'$ is well-constrained, so is $P$. And since a reduction relies on straightedge and compass construction, we have: if $P \mapsto P'$, and if $P'$ is **S**, so is $P$. Therefore, if $P \mapsto P'$ and $P$ is **U**, $P'$ is **U** or **R** or **L**. If $P$ is **L** it can be reduced to **R** or **L** problem; if it is **R**, it can be reduced only to another **R** problem.

*Example* 4.

- The problem 4: $(A, B, G)$ **S** from Wernick's list can be reduced to the problem 2: $(A, B, M_a)$ **S**, so it holds: $(A, B, G) \mapsto (A, B, M_a)$. The opposite is true since the point $G$ can be constructed from points $A$ and $M_a$.
- The problem 55: $(A, H, T_a)$ **S** can be reduced to the problem 56: $(A, H, T_b)$. Indeed, since the problem 55 is solvable, there is a construction of $(A, B, C)$ and, further, of $(A, H, T_b)$. Note, however, that the problem 56 is **U**, and that there is no reduction from problem 56 to problem 55. We can see in this example, and the following ones, that the relation $\mapsto$ is not symmetric.
- The problem 7: $(A, B, H)$ **S** can be reduced to the problem 6: $(A, B, H_c)$ **L**, but the opposite is false.
- Every **R** problem leads to a simple way to reduce some problems. For instance, the problem 71: $(O, G, H)$ is **R** because of Euler relation $\overrightarrow{OH} = \overrightarrow{OG}$. Then, if two of these three points $(O, G, H)$ appear in a problem, one of them can be replaced by the third. This way, the problems 72, 79 and 120 are mutually reducible to each other.

**Definition 6.** A subset $\mathcal{C}$ of a set of construction problems $\mathcal{S}$ is $\mathcal{S}$*-complete* if (Figure 3):

- for each constructible problem $P$ from $\mathcal{S}$, there is a problem $P' \in \mathcal{C}$, such that $P \mapsto P'$;
- for each unconstructible problem $P$ from $\mathcal{S}$, there is a problem $P' \in \mathcal{C}$, such that $P' \mapsto P$.

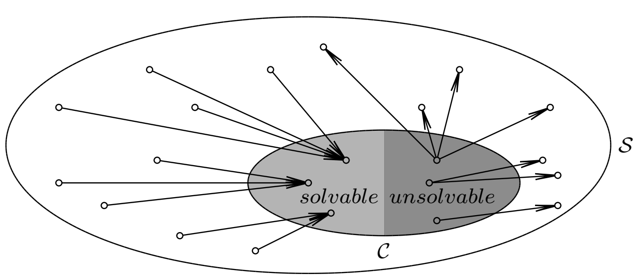


FIGURE 3. Illustration of the notion of $\mathcal{S}$-complete sets

Given a set of construction problems $\mathcal{S}$, our goal is to find a small (in terms of cardinality) subset $\mathcal{C}$ of $\mathcal{S}$ such that $\mathcal{C}$ is $\mathcal{S}$-complete. This would lead to a small set of "key" problems from $\mathcal{S}$

and to a separation of a, hopefully simple, geometry knowledge needed for reductions and a knowledge needed for establishing constructibility. That would enable simpler justification of resolvability of problems, simpler presentation (for educational purposes – focusing on the "key" problems), and automation of the solving process. Very importantly, algebraic arguments for proving unconstructibility would be required only for a small number of problems, while unconstructibility for other unconstructible problems would be proved using means of synthetic geometry.

For detecting some $\mathcal{S}$-complete sets of problems, we need some effective, hopefully simple, way for checking if two sets of points are in the $\mapsto$ relation. Instead of checking all possible straightedge-and-compass constructions, we will consider simple production rules that link small sets of points with points that can be constructed from them.

**Definition 7 (Production rule).** If, given a set of characteristic triangle points $S$, a characteristic point $X$ is constructible, we say that there is *a production rule* $S \to X$.

**Definition 8 (Relation $\Rightarrow_R$).** Given a set $R$ of production rules and a set of points $\mathcal{X}$, if there is a rule $S \to X$ in $R$, such that $S\sigma \subseteq \mathcal{X}$, and $X\sigma \notin \mathcal{X}$, then it holds $\mathcal{X} \Rightarrow_R \mathcal{X} \cup \{X\sigma\}$, where $\sigma$ is a permutation of triangle vertices (that is applied as a substitution simultaneously).

*Example* 5. Given a set of production rules $R = \{\{A, B, H\} \to H_c\}$ and a set $\mathcal{X}$ of points $\{A, C, H, G\}$, then it holds $\{A, C, H, G\} \Rightarrow_R \{A, C, H, G, H_b\}$ (where the substitution $\sigma$ maps $A$ to $A$, $C$ to $B$, $B$ to $C$ and, hence, $H_c$ to $H_b$).

**Definition 9 (Relation $\Rightarrow_R^*$).** The relation $\Rightarrow_R^*$ is the reflexive and transitive closure of the relation $\Rightarrow_R$.

Obviously, if $S \Rightarrow_R^* S'$, then the set $S'$ is constructible from the set $S$. The following theorem is a trivial consequence.

**Theorem 3.1.** *Given a set $R$ of production rules and triangle location problems $P$ and $P'$ if $P \Rightarrow_R^* P'$ then $P \mapsto P'$.*

Of course, the converse of the above theorem does not hold in general (for arbitrary $R$). The characterization of sufficient conditions $P \mapsto P'$ given by the above theorem is defined modulo a set of rules $R$. If $R$ is empty or small, then it is not very helpful in using Theorem 3.1. For detecting $\mathcal{S}$-complete sets for a concrete set $\mathcal{S}$, we have to consider suitable production rules and to look for a simple yet rich enough set $R$.

Thanks to Theorem 3.1, with a suitable set $R$ of production rules, the reduction phase (in showing the relation $\mapsto$ for two sets) boils down to a trivial syntactical procedure that does not need any geometrical knowledge.

In this paper we focus on finding small sets of Wernick-complete problems (in the next section), but the same approach can be used for other sets of construction problems.

## 4. Wernick-complete Sets of Problems

For a set of production rules $R$, by Theorem 3.1, given two Wernick's problems $P$ and $P'$ it holds $P \mapsto P'$ if $P \Rightarrow_R^* P'$. Therefore, a set of construction problems $\mathcal{C}$ is Wernick-complete if it holds:

- for each **S** Wernick's problem $P$, there is a problem $P' \in \mathcal{C}$, such that $P \Rightarrow_R^* P'$;
- for each **U** Wernick's problem $P$, there is a problem $P' \in \mathcal{C}$, such that $P' \Rightarrow_R^* P$.

### 4.1. Production Rules

For building a set of production rules for Wernick's points, we used our tool ArgoTriCS for automated solving of construction problems [24, 22]. The tool, using its knowledge base (consisting of definitions, lemmas and construction primitives), can solve almost all solvable problems from Wernick's list, but here we are not interested in these solutions, but in production rules that the tool can yield. Recall that our goal is not only justifying constructibility for solvable problems, but also justifying unconstructibility for unsolvable problems.

By default, ArgoTriCS starts with given triple of Wernick's points and tries, using the available geometry knowledge and available construction primitives, to construct vertices $A$, $B$, and $C$. For the purpose of our goal, we slightly modified the tool, so it can take any number of Wernick's points and then try to construct all other Wernick's points. We systematically used the following procedure, for $k = 2, 3, 4$: take $k$ of Wernick's points and try to construct any of other $16 - k$ Wernick's points. For $k = 2$, there are $\binom{16}{2} = 120$ such runs, for $k = 3$, there are $\binom{16}{3} = 560$ such runs, and for $k = 4$, there are $\binom{16}{4} = 1820$ such runs. We eliminate each rule that is subsumed by some other rule: the $S \to X$ is subsumed by $S' \to X$ if $S' \subset S$. With the available knowledge of ArgoTriCS, we ended up with:

- for $k = 2$: 21 rules (for example, $\{A, B\} \to M_c$),
- for $k = 3$: 3324 rules (for example, $\{A, G, H\} \to I$),
- for $k = 4$: 897 rules (for example, $\{A, H, T_a, I\} \to B$),

altogether 4242 rules in our initial set of rules $R$. Note that the rule $\{A, H, T_a, I\} \to B$ is indeed not subsumed by some rule obtained for $k = 3$. Namely, the problems $\{A, H, T_a\}$ and $\{A, H, I\}$ are **S** (so, from these triples, the point $B$ can be obtained), but ArgoTriCS cannot solve them (so, it does not have all relevant geometry knowledge). The problems $\{A, T_a, I\}$ and $\{H, T_a, I\}$ are, respectively, **L** and **U** and from these triples the point $B$ cannot be constructed using the given set of rules.

### 4.2. Reductions Detected

We used the above set $R$ of production rules to detect pairs of problems $(P, P')$. We identified 4745 reductions from **S** problems to **S** problems (involving 161 production rules), and 18 reductions from **U** problems to **U** problems (involving 6 production rules). It is interesting to note that, after using all production rules with $k = 2$ and $k = 3$, no new reductions were identified for the rules with $k = 4$.

With a set of reductions detected, say from **S** to **S** problems, we need to find a smallest set $\mathcal{S}$ (in the sense of cardinality) such that each **S** problem can be reduced to some problem from $\mathcal{S}$ (including, possibly, to itself). This problem is actually the set covering problem (SCP), one of classical NP-complete problems, shown to be NP-complete in 1972 [14]. SCP problem can be formulated as follows: given a set $\mathcal{U}$ of elements $\{1, 2, ..., m\}$ and a set $\mathcal{P}$ of $n$ sets whose union equals $\mathcal{U}$, the problem is to identify the smallest subset of $\mathcal{P}$ whose union equals $\mathcal{U}$. In our context, $\mathcal{U}$ is the set of all **S** problems. For each $\mathcal{S}$ problem, a set of all **S** problems that can be reduced to it (including that problem itself) belongs to $\mathcal{P}$ (and $\mathcal{P}$ consists of such sets only). Finding the smallest subset of $\mathcal{P}$ whose union equals $\mathcal{U}$ is thus a problem of finding the smallest set of **S** problems, "key problems", such that any **S** problem can be reduced to one of the "key problems". For solving relevant instances of SCP, we used our SAT-based constraint solver URSA [18], with specifications automatically generated with the help of ArgoTriCS and additional custom tools.

The URSA system revealed a Wernick-complete set $\mathcal{S}$ of problems consisting of:

- 9 **S** problems: 55, 57, 69, 76, 82, 87, 108, 119, 131;
- 33 **U** problems: 56, 68, 72, 74, 77, 78, 81, 88, 89, 90, 95, 100, 106, 107, 109, 110, 111, 113, 114, 115, 118, 121, 122, 123, 127, 128, 130, 132, 134, 135, 136, 137, 138.

The results for **S** problems were not surprising: each problem solvable by ArgoTriCS can be, using the rules from $R$, reduced to $(A, B, C)$ and then, further to any other problem. For this, 127 production rules are used. This also shows that the ArgoTriCS solving mechanism can be replaced by a solving mechanism that uses only production rules. If all **S** problems were solvable by ArgoTriCS, then in $\mathcal{S}$ there could be just one problem. However, ArgoTriCS cannot solve all **S** problems – it cannot solve exactly the nine problems listed above (and any other problem can be reduced to any of it).

The results for **U** problems were a bit disappointing: only to six **U** problems can be reduced some other **U** problem. For this, 3 production rules are used.

Altogether, we identified a Wernick-complete set of problems consisting of 42 problems and status of each Wernick's $P$ problem can be determined in the following way:

- Check if the problem is **R**;
- If not, check if the problem is **L**;
- If not, check if there is a **S** problem $P' \in \mathcal{C}$ such that $P \Rightarrow_R^* P'$; if *yes*, $P$ is **S**;
- If not, check if there is a **U** problem $P' \in \mathcal{C}$ such that $P' \Rightarrow_R^* P$; if *yes*, $P$ is **U**;

using the set $R$ of 128 production rules (all of the rules needed for checking if the problem is **R** are contained in the set of 127 rules needed for **S** problems, and only one from three rules needed for **U** problems was not listed in these 127 rules). It was shown that for checking if the problem is **R**, one needs only to use 3 production rules: $\{A, B\} \to M_c$, $\{A, M_a\} \to G$, and $\{O, G\} \to H$. For checking if the problem is **L**, one has to use another inference rules, not only production rules.

*Example* 6. Let's take a look at few examples from Wernick's list.

- The status of the problem 3: $(A, B, M_c)$ is determined to be **R**, since there is a production rule $\{A, B\} \to M_c$ in the set of rules identified by the ArgoTriCS system;
- The problem 1: $(A, B, O)$ is determined to be **L**, since it was not determined to be **R** and ArgoTriCS identified that the point $O$ has to belong to the bisector of the segment $AB$ in order for problem to have a solution;
- The problem 24: $(A, M_a, H)$ was not found to be **R** nor **L**, but for the set of the rules $R = \{\{A, M_a\} \to G, \{M_a, G, H\} \to I\}$ it can be reduced to one of the core **S** problems – the problem 57: $(A, H, I)$. Therefore it is **S**;
- The problem 73: $(O, G, I)$ is marked **U** since it does not satisfy neither of first three items of the procedure described above, and the problem 121: $(G, H, I)$, which is one of the core **U** problems, can be reduced to it (using the rule $\{G, H\} \to O$).

### 4.3. Small Sets of Rules, Small Sets of Complete Problems

Above we described the Wernick-complete set of 42 problems, accompanied with 128 production rules. We would like to have smaller number of rules. However, there is an obvious trade-off: the smaller number of rules, the bigger the number of problems in the Wernick-complete set.

We try to identify a small set of "the most important production rules". We put together all the rules ($\{O, G\} \to H$, $\{G, H\} \to O$, and $\{M_a, G\} \to A$) needed for **U** reductions and all the rules ($\{A, B\} \to M_c$, $\{A, M_a\} \to G$, and $\{O, G\} \to H$) needed for showing that a problem is **R**. With this set of only 5 distinct production rules, we identify 51 reductions from **S** to **S** problem, but the Wernick-complete set consists of 58 **S** problems, which in total gives 33+58=91 problems in Wernick-complete set (there are 33 **U** problems in Wernick-complete set).

Afterwards, we try to add the rules that are most frequently used and to see if using them more reductions can be found, bringing along a smaller set of Wernick's complete problems. This approach did not bring much success (as we hoped), some frequent rules brought many new reductions, and some none at all. We ended up with 8 production rules, 86 reductions from **S** to **S** problem, and the set of 33+56=89 problems in the Wernick-complete set.

We continue in extending the set of rules (and reducing the number of problems in Wernick-complete set of problems) by using the next heuristic: for problems that could not be reduced to some other problem using the current set of rules, we detected what are the problems that the considered problem is reduced into using the smallest number of rules. Then these rules are added to the current set of rules. We came to the set of 36 production rules and the Wernick-complete set consisting of 36+33 problems. In Figure 4, a graphical representation of the reductions found is given – each reduction is represented by one arrow, where black arrows correspond to reductions between **S** problems, while gray arrows correspond to reductions between **U** problems.

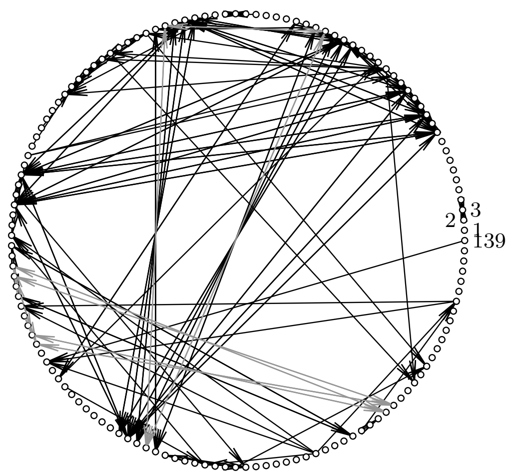


FIGURE 4. Graphical representation of reductions found

In the following text, we illustrate a "standard construction" (generated automatically by ArgoTriCS tool) and a reduction-based construction described in terms of production rules.

*Example* 7. For the problem 11: $(A, O, M_a)$ from Wernick's list the construction generated by ArgoTriCS is:

1. Using the point $A$ and the point $M_a$, construct a point $G$ (rule W01);
2. Using the point $O$ and the point $G$, construct a point $H$ (rule W01);
3. Using the point $A$ and the point $H$, construct a line $h_a$ (rule W02); % DET: points $A$ and $H$ are not the same;
4. Using the point $A$ and the point $O$, construct a circle $k(O, C)$ (rule W06); % NDG: points $A$ and $O$ are not the same;
5. Using the point $M_a$ and the line $h_a$, construct a line $a$ (rule W10a);
6. Using the circle $k(O, C)$ and the line $a$, construct a point $C$ and a point $B$ (rule W04); % NDG: line $a$ and circle $k(O, C)$ intersect.

On the other hand, by using the production rules:

$$
\begin{aligned}
R = \{&R_1 = \{A, M_a\} \to G, R_2 = \{O, G\} \to H, R_3 = \{M_a, G, H\} \to T_b,\\
&R_4 = \{M_a, G, H\} \to T_c, R_5 = \{M_a, G, H\} \to I, R_6 = \{T_b, T_c, I\} \to M_b,\\
&R_7 = \{M_b, G\} \to B, R_8 = \{B, M_a\} \to C\}
\end{aligned}
$$

a following construction is obtained (in this case, different than the one generated by ArgoTriCS):

$$
\begin{aligned}
\{A, O, M_a\} &\to_{R_1} \{A, O, M_a, G\}\\
&\to_{R_2} \{A, O, M_a, G, H\}\\
&\to_{R_3} \{A, O, M_a, G, H, T_b\}\\
&\to_{R_4} \{A, O, M_a, G, H, T_b, T_c\}\\
&\to_{R_5} \{A, O, M_a, G, H, T_b, T_c, I\}\\
&\to_{R_6} \{A, O, M_a, G, H, T_b, T_c, I, M_b\}\\
&\to_{R_7} \{A, O, M_a, G, H, T_b, T_c, I, M_b, B\}\\
&\to_{R_8} \{A, O, M_a, G, H, T_b, T_c, I, M_b, B, C\}
\end{aligned}
$$

Since the points $A$, $B$, and $C$ are constructed, the problem is solved.

## 5. Conclusions and Future Work

In this paper, we make explicit the notion of problem reduction for triangle location problems. The notion of reduction is widely used in theoretical computer science but, as far as we know, it was never formalized this way. We use these techniques within Wernick's corpus to try to discover related construction and to reduce unconstructible problems to a few number of them whose unconstructibility is assumed. Our hope was to avoid the need of advanced algebraic tools in proving unconstructibility. We found only few reductions for **U** problems, but much more for **S** problems. Notice that for a **S** problem $P$ reduced to some of the key problems, we also get a concrete construction for $P$. With additional knowledge useful for solving **S** problems, there might be additional reductions also between **U** problems.

This study suggests tracks for future researches about solving geometric construction problems by reduction. Firstly, we will study other corpora of triangle location problems, such as the one proposed by Connelly [5], where the relations between characteristic points are more dense. Moreover, in a greater corpus it should be possible to find more reductions.

Another interesting point lies in the relation $\Rightarrow_R^*$ based on a set of production rules from high school. The relation $\Rightarrow_R^*$ is weaker than $\mapsto$ and in constrast to the latter, it puts a structure on the set of **S** problems. A good choice for the set $R$ should define an order on the **S** problems making a hierarchy able to simplify some constructions. Finally, the notion of reduction has to be generalized for tacking more corpora into account and to allow a finer analysis of the subtended geometry. We are also planning to extend the notions defined in this paper to all other sorts of construction problems.

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