Wernick's List: A Final Update

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Abstract. We present a final status of all problems from Wernick's list of triangle construction problems published in 1982 and with a number of unknown status until recently. Our results were obtained by a computer-based system for checking constructibility. We also developed a system for finding elegant constructions for solvable problems and for verifying their correctness. These systems helped in resolving problems open for decades, showing the power of modern computer systems in areas such as symbolic computation, problem solving, and theorem proving.

1 Introduction

In 1982, Wernick presented a list of straightedge and compass construction problems [23] (many of these problems were considered along the centuries, before this list was compiled). Each of them is a *triangle location problem*: the task is to construct a triangle ABC starting from three located points selected from the following set of sixteen characteristic points:

- A, B, C, O: three vertices and circumcenter;
- M_a, M_b, M_c, G: the side midpoints and centroid;
- H_a, H_b, H_c, H: three feet of altitudes and orthocenter;
- T_a, T_b, T_c, I: three feet of the internal angles bisectors and incenter.

There are 560 triples of the above points, but Wernick's list consists only of 139 significantly different non-trivial problems. The triple $\{A, B, C\}$ is trivial and, for instance, the problems $\{A, B, M_a\}$, $\{A, B, M_b\}$, $\{B, C, M_b\}$, $\{B, C, M_c\}$, $\{A, C, M_a\}$, and $\{A, C, M_c\}$ are considered to be symmetric (i.e., analogous). Wernick divided the problems into four categories:

Redundant problems: if there is a point in the given triple such that it is uniquely determined and constructible from the remaining two points, we say that the problem is *redundant* (and we denote it by **R**). For instance, the triple $\{A, B, M_c\}$ is redundant — given points A and B, the point M_c is uniquely determined.

Locus dependent problems: if there exists the required triangle ABC (not a way to construct it, but the triangle itself) only for given points meeting certain constraints, then we say that the problem is *locus dependent* (and we denote it by **L**). All such problems in Wernick's list have infinitely many solutions. For instance, for the problem $\{A, B, O\}$, the point O has to belong to the perpendicular bisector of AB, otherwise the corresponding triangle ABC does not exist.

Solvable problems: if there is a construction of the required triangle ABC (whenever it exists, while it does not exist only in some special cases) starting with the given points, we say that the problem is *solvable* or *constructible* (and we denote it by **S**).

Unsolvable problems: if for some given points the required triangle ABC exists, but it is not constructible, then we say that the problem is *unsolvable* or *unconstructible* (and we denote it by **U**).

1. A, B, O	L	36. A, M_b, T_c	S	71. O, G, H	R	106. M_a, H_b, T_c	U [18]
2. A, B, M_a	S	37. A, M_b, I	S	72. O, G, T_a	U [18]	107. M_a, H_b, I	U [18]
3. A, B, M_c	R	38. A, G, H_a	L	73. O, G, I	U [18]	108. M_a, H, T_a	**S** [20]
4. A, B, G	S	39. A, G, H_b	S	74. O, H_a, H_b	U [18]	109. M_a, H, T_b	U [21]
5. A, B, H_a	L	40. A, G, H	S	75. O, H_a, H	S	110. M_a, H, I	U [21]
6. A, B, H_c	L	41. A, G, T_a	S	76. O, H_a, T_a	S	111. M_a, T_a, T_b	U [21]
7. A, B, H	S	42. A, G, T_b	U [18]	77. O, H_a, T_b	U [20]	112. M_a, T_a, I	S
8. A, B, T_a	S	43. A, G, I	S [18]	78. O, H_a, I	U [20]	113. M_a, T_b, T_c	U [20]
9. A, B, T_c	L	44. A, H_a, H_b	S	79. O, H, T_a	U [18]	114. M_a, T_b, I	U [18]
10. A, B, I	S	45. A, H_a, H	L	80. O, H, I	U [18]	115. G, H_a, H_b	U [18]
11. A, O, M_a	S	46. A, H_a, T_a	L	81. O, T_a, T_b	U [20]	116. G, H_a, H	S
12. A, O, M_b	L	47. A, H_a, T_b	S	82. O, T_a, I	S [18]	117. G, H_a, T_a	S
13. A, O, G	S	48. A, H_a, I	S	83. M_a, M_b, M_c	S	118. G, H_a, T_b	U [20]
14. A, O, H_a	S	49. A, H_b, H_c	S	84. M_a, M_b, G	S	119. G, H_a, I	S [20]
15. A, O, H_b	S	50. A, H_b, H	L	85. M_a, M_b, H_a	S	120. G, H, T_a	U [18]
16. A, O, H	S	51. A, H_b, T_a	S	86. M_a, M_b, H_c	S	121. G, H, I	U [18]
17. A, O, T_a	S	52. A, H_b, T_b	L	87. M_a, M_b, H	S [18]	122. G, T_a, T_b	U [20]
18. A, O, T_b	S	53. A, H_b, T_c	S	88. M_a, M_b, T_a	U [18]	123. G, T_a, I	U [20]
19. A, O, I	S	54. A, H_b, I	S	89. M_a, M_b, T_c	U [18]	124. H_a, H_b, H_c	S
20. A, M_a, M_b	S	55. A, H, T_a	S	90. M_a, M_b, I	U [21]	125. H_a, H_b, H	S
21. A, M_a, G	R	56. A, H, T_b	U [18]	91. M_a, G, H_a	L	126. H_a, H_b, T_a	S
22. A, M_a, H_a	L	57. A, H, I	S [18]	92. M_a, G, H_b	S	127. H_a, H_b, T_c	U [20]
23. A, M_a, H_b	S	58. A, T_a, T_b	S [18]	93. M_a, G, H	S	128. H_a, H_b, I	U [20]
24. A, M_a, H	S	59. A, T_a, I	L	94. M_a, G, T_a	S	129. H_a, H, T_a	L
25. A, M_a, T_a	S	60. A, T_b, T_c	S	95. M_a, G, T_b	U [18]	130. H_a, H, T_b	U [18]
26. A, M_a, T_b	U [18]	61. A, T_b, I	S	96. M_a, G, I	S [18]	131. H_a, H, I	S [18]
27. A, M_a, I	S [18]	62. O, M_a, M_b	S	97. M_a, H_a, H_b	S	132. H_a, T_a, T_b	U [20]
28. A, M_b, M_c	S	63. O, M_a, G	S	98. M_a, H_a, H	L	133. H_a, T_a, I	S
29. A, M_b, G	S	64. O, M_a, H_a	L	99. M_a, H_a, T_a	L	134. H_a, T_b, T_c	U [20]
30. A, M_b, H_a	L	65. O, M_a, H_b	S	100. M_a, H_a, T_b	U [18]	135. H_a, T_b, I	U [20]
31. A, M_b, H_b	L	66. O, M_a, H	S	101. M_a, H_a, I	S	136. H, T_a, T_b	U [20]
32. A, M_b, H_c	L	67. O, M_a, T_a	L	102. M_a, H_b, H_c	L	137. H, T_a, I	U [20]
33. A, M_b, H	S	68. O, M_a, T_b	U [18]	103. M_a, H_b, H	S	138. T_a, T_b, T_c	U [22]
34. A, M_b, T_a	S	69. O, M_a, I	S	104. M_a, H_b, T_a	S	139. T_a, T_b, I	S
35. A, M_b, T_b	L	70. O, G, H_a	S	105. M_a, H_b, T_b	S		

Table 1: The definite status of all Wernick's problems

In the original list, the problem 102 was erroneously marked **S** instead of **L** [18] and the problem 108 was erroneously marked **U** instead of **S** [20]. Wernick's list left 41 problem unresolved/unclassified, but the update from 1996 [18] left only 20 of them. In the meanwhile, the problems 90, 109, 110, 111 [21], and 138 [22] were proved to be unsolvable. We are not aware of published solutions for remaining 15 unsolved problems (although there are indications that eight more were resolved in the meanwhile [25]). Some of the problems were additionally considered for simpler solutions, like the problem 43 [1, 5], the problem 57 [24], and the problem 58 [4, 21]. Solutions for 59 solvable problems can be found on the Internet [21]. The status for all these problems was determined by ad-hoc attempts, with no systematic solving procedures or computer support involved.

Recently, we developed computer-based systems for checking constructibility for all problems from Wernick's list [20] and for finding constructions for solvable problems [16, 17, 13]. Thanks to the former system, we were able to fill-in all remaining slots in Wernick's list and now the status for all 139 problems is known. They are given in Table 1: there are 74 **S** problems, 39 **U** problems, 3 **R** problems, and 23 **L** problems. The problems are associated with references to the papers resolving their status (for the problems with no references, the status was given in the original Wernick's paper). More on these two systems is given in the following two sections.

2 Computer-Assisted Resolving of Unconstructible Problems

Our first method relies on algebraization of geometric constructions and Galois results about straightedge and compass constructions of numbers. Let us first recall some classical results.

Let F be a field extension of $\mathbb{Q}$, and G a field extension of F. A number in G is straightedge and compass constructible in F if and only if it is equal to an expression using only numbers in F, arithmetic operations and square radicals. Such a number is algebraic in F, and its degree over F is a power of two. This result is known as Wantzel's result and is often used to prove that a number is not straightedge and compass constructible (for instance, in the demonstration of the impossibility of angle trisection using only straightedge and compass). The conjecture which states the opposite direction is generally false. This is why we also use a stronger result which is a consequence of Galois theory: an algebraic number on F is constructible if and only if the splitting field of its minimal polynomial is an extension of degree 2^m for some m over F. This is equivalent to the fact that the cardinal of the Galois group of the minimal polynomial is also 2^m.

A point is straightedge and compass constructible from a set $\mathcal{B}$ of points if its coordinates are constructible on the extension of $\mathbb{Q}$ containing the coordinates of the points of $\mathcal{B}$. It is obvious that one of the points from $\mathcal{B}$ can have coordinates $(0, 0)$, and another one can have coordinates $(k, 0)$ where k is a given number. With Wernick's corpus, $\mathcal{B}$ contains three points, two of them can be fixed this way, whereas the third one must have free coordinates (a, b) in order to consider the generic case.

Let us also give a more precise meaning of the labels annotating the problems in Wernick's corpus. A problem has status **S** or **U** if it has solutions in the Euclidean plane, regardless constructibility using straightedge and compass: it has label **S** if it is straightedge and compass constructible, and label **U** (unconstructible) otherwise. The labels **R** and **L** correspond to over-constrained problems and are easy to check by using algebraic tools. We will not discuss this matter further within this text.

The general idea of the method consists of the following steps:

- translate the considered problem into a polynomial system,
- use *regular chains* to obtain a disjunction of irreducible polynomial systems,
- use Wantzel's result or Galois theory to prove constructibility or unconstructibility.

We made this pipeline automatic through an implementation in Maple [11] which offers several powerful tools like the regular chains and the computation of Galois group of a polynomial up to degree 9.

Actually, this idea is used in two different ways:

- First, we try to prove that the problem is not constructible: for this, we consider a *witness*, that is an example of triangle which is a solution of an instance of the problem with rational coordinates for the given points and we apply the method to this example. If this example is not constructible, then the problem is not solvable by straightedge and compass. We implemented a routine for automatically producing witness candidates and checking the whole list for unconstructibility.
- If the first method fails to prove the unconstructibility of the problem (for several witness candidates), we apply the method on the *parametric problem* which represents the general case. The calculi are much harder but complete: if each Galois group has a power of 2 as order, then the problem is constructible. And then, it is theoretically possible to extract a construction [2, 8], but it is very difficult to obtain and even for the simplest problems, the generic construction is geometrically unappealing. See, for instance, the problem 108 below.

Example 1. We prove the unconstructibility of the problem 122: $\{G,\ T_a,\ T_b\}$ by choosing the coordinates $T_b(0,0)$, $T_a(4,0)$ and $G(2,1)$. Each of these points gives rise to two polynomial equations involving coordinates of points $A(x_A, y_A)$, $B(x_B, y_B)$, and $C(x_C, y_C)$. The triangularization process for this system of 6 equations gives two systems containing the following disqualifying equations:
$P(y_C) = y_C^4 - 6y_C^3 - 51y_C^2 - 24y_C + 36$
the splitting field of which is of order 24, meaning that even if the degree of the polynomial is 4, it is not solvable by square radicals, and
$P(y_C) = 2701y_C^3 - 12871y_C^2 + 43008y_C - 28224$
with degree 3. Therefore, this problem is not constructible.

Example 2. In the problem 108, the given points are T_a, H and M_a, we use the coordinates $(0,0)$ for T_a, $(1,0)$ for M_a, and parametric coordinates (a,b) for H. The triangularization of the corresponding polynomial system gives the following system:

$$\begin{cases} x_C + x_B - 2 = 0 \\ -a^2 - by_A + x_B^2 + 2a - 2x_B = 0 \\ y_C = 0 \\ y_B = 0 \\ x_A - a = 0 \\ a^3 + aby_A - a^2 + y_A^2 = 0 \end{cases}$$

which is obviously constructible (all the equations have degree at most 2) and moreover, it is simple enough to solve with square radicals, for instance $y_A = (a/2)(-b \pm \sqrt{b^2 - 4a + 4})$, and to translate the formulas into a straightedge and compass construction that mimics the computation (Figure 1). Recall that it is possible to perform additions, multiplications, divisions and root extract by using ruler and compass constructions.

This construction might not be elegant, but it is perfectly valid. A new challenge might be to find appealing geometric constructions for problems 108[3] and 119 (see below).

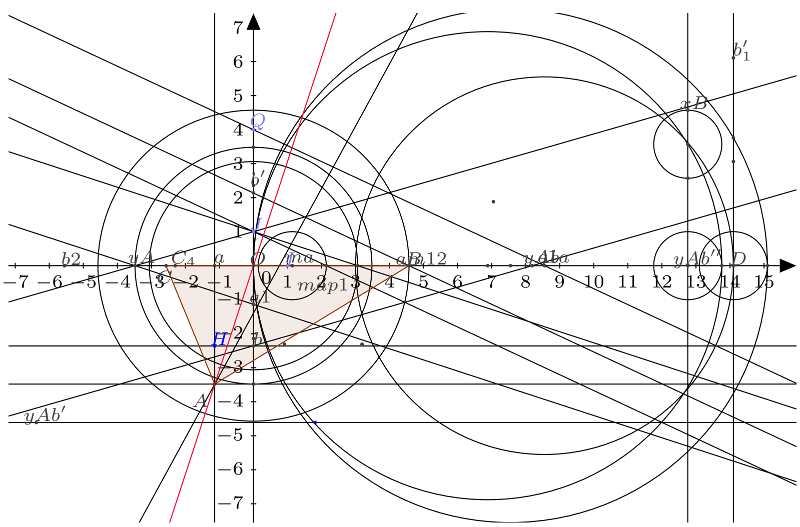


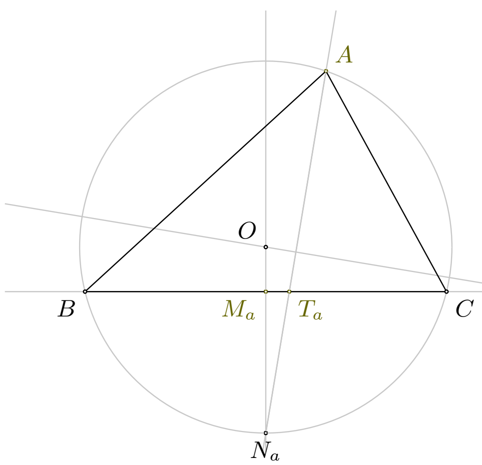
Fig. 1: Geometric translation in GeoGebra of the system given in Example 2. Parameters a and b correspond to the *free* point H: this point can be moved and the figure is transformed accordingly.

In Appendix, we list relevant polynomials for all the problems with unknown status [20].

[3] The GeoGebra figure can be found at url `https://sites.google.com/site/pascalschreck/adg14`

3 Computer-Assisted Solving of Constructible Problems

Our second system, ArgoTriCS, equipped with relevant geometry knowledge, pursues very different aims. It is capable of solving almost all solvable problems from Wernick's list: 66 out of 74 [16, 15, 13]. The system was implemented in PROLOG, has around 6000 lines of code, while the solving times span from a couple of milliseconds to more than an hour. The longest generated construction is the construction for the problem 101: $\{M_a, H_a, I\}$ – it consists of 14 steps (mostly compound construction steps, such as construction of the midpoint of a segment). The system also detects if the problem is redundant or locus dependent. The system produces a construction in a natural language form, and in the format of a dynamic geometry tool GCLC [9], so corresponding illustrations can be also automatically generated. The next example shows an automatically generated solution for the problem 25 : $\{A, M_a, T_a\}$ (along with non-degenerate conditions and determination conditions), while the corresponding illustration is given in Figure 2.



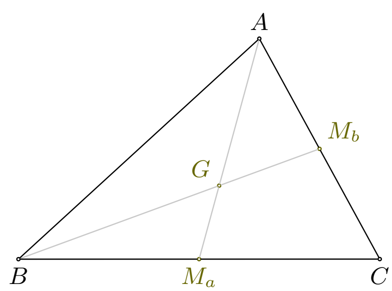


Fig. 2: Illustration for the problem 25 (left) and for the problem 84 (right)

Example 3. Given points A, M_a, and T_a, construct the triangle ABC.

1. *Using the point A and the point T_a, construct a line s_a (rule W02);*
2. *Using the point M_a and the point T_a, construct a line a (rule W02);*
3. *Using the point M_a and the line a, construct a line m_a (rule W10b);*
4. *Using the line m_a and the line s_a, construct a point N_a (rule W03);*
5. *Using the point A and the point N_a, construct a line $m(AN_a)$ (rule W14);*
6. *Using the line $m(AN_a)$ and the line m_a, construct a point O (rule W03);*

7. *Using the point A and the point O, construct a circle $k(O, C)$ (rule W06);*
8. *Using the circle $k(O, C)$ and the line a, construct a point C and a point B (rule W04).*

Non-degenerate conditions: line a and circle $k(O, C)$ intersect; points A and O are not the same; lines $m(AN_a)$ and m_a are not parallel; lines m_a and s_a are not parallel.

Determination conditions: lines $m(AN_a)$ and m_a are not the same; points A and N_a are not the same; lines m_a and s_a are not the same; points M_a and T_a are not the same; points A and T_a are not the same.

Unlike other systems for automatically solving construction problems, ArgoTriCS also considers correctness of the constructions generated and invokes automated geometry theorem provers – OpenGeoProver [12] and the provers built in the GCLC tool. Each construction generates three theorems – one for each given point; for instance, if the point G is given, then it should be proved that G is indeed the centroid of the constructed triangle ABC. So, for 92 problems solved by ArgoTriCS (66 **S** problems, and all **L** and **R** problems), there are 276 theorems (some of them trivial – if a triangle vertex is given). Out of 276 theorems, 194 were successfully proved by at least one prover. In addition, for all problems involving only the points $A, B, C, M_a, M_b, M_c, G$, we generated machine verifiable proofs for the correctness of constructions – proofs verified by the proof assistant Isabelle [19]. The next example gives an automatically generated solution for the problem $84 : \{M_a, M_b, G\}$, illustrated in Figure 2.

Example 4. Given points M_a, M_b, and G, construct the triangle ABC.

1. *Using the point M_a and the point G, construct a point A (rule W01);*
2. *Using the point M_b and the point G, construct a point B (rule W01);*
3. *Using the point M_a and the point B, construct a point C (rule W01).*

No non-degenerate conditions.
No determination conditions.

For this problem, the central theorem proved formally within the Isabelle proof assistant, with a help of automated theorem provers, is the following:

$$\begin{aligned}&\forall M_a, M_b, G.\\&\neg collinear(M_a, M_b, G) \Leftrightarrow \exists A, B, C.(midpoint(M_a, B, C)\wedge\\&midpoint(M_b, A, C) \wedge centroid(G, A, B, C) \wedge \neg collinear(A, B, C))\end{aligned}$$

The system ArgoTriCS was used for automatically generating a compendium[4] of all problems from the extended Wernick's list (for all 560 triples of characteristic problems) – spanning around 3000 pages, and also an on line encyclopedia with animated solutions for all solved problems [14].

[4] Available online from: `http://www.matf.bg.ac.rs/~vesnap/compendium_wernick.pdf`

4 Conclusions and Future Work

In this paper we presented the final version of Wernick's list – a list of triangle location problems, presented in 1982 and with a number of construction problems with unknown statuses until recently. These updates were produced by our computer-based systems, while for almost all solvable problems our system can produce elegant constructions with associated illustrations. These results show the power of modern computer systems in areas such as symbolic computation, problem solving and theorem proving.

For future work, we are planning to consider, in analogy, other corpora of triangle construction problems — location problems involving additional points [3] or construction problems based on various geometrical quantities [10, 6, 7].

References

1. J. Anglesio and V. Schindler. Solution to problem 10719. *American Mathematical Monthly*, 107:952–954, 2000.
2. Guoting Chen. Les constructions à la règle et au compas par une méthode algébrique (in french). Master's thesis, Université Louis Pasteur, 1992. available at url https://sites.google.com/site/pascalschreck/home/recherche/arti.
3. Harold Connelly. An Extension of Triangle Constructions from Located Points. *Forum Geometricorum*, 9:109–112, 2009.
4. Harold Connelly, Nikolaos Dergiades, and Jean-Pierre Ehrmann. Construction of Triangle from a Vertex and the Feet of Two Angle Bisectors. *Forum Geometricorum*, 7:103–106, 2007.
5. Eric Danneels. A Simple Construction of a Triangle from its Centroid, Incenter, and a Vertex. *Forum Geometricorum*, 5:53–56, 2005.
6. V. B. Fursenko. Lexicographic account of triangle construction problems (part I). *Mathematics in schools*, 5:4–30, 1937. In Russian.
7. V. B. Fursenko. Lexicographic account of triangle construction problems (part II). *Mathematics in schools*, 6:21–45, 1937. In Russian.
8. Xiao-Shan Gao and Shang-Ching Chou. Solving geometric constraint systems. II. A symbolic approach and decision of Rc-constructibility. *Computer Aided Design*, 30(2):115–122, 1998.
9. Predrag Janičić. Geometry Constructions Language. *Journal of Automated Reasoning*, 44(1-2):3–24, 2010.
10. L. Lopes. *Manuel de Construction de Triangles*. QED Texte, 1996.
11. Maplesoft. Maple 18. Maplesoft, a division of Waterloo Maple Inc., Waterloo, Ontario.
12. Filip Marić, Ivan Petrović, Danijela Petrović, and Predrag Janičić. Formalization and implementation of algebraic methods in geometry. In Pedro Quaresma and Ralph-Johan Back, editors, Proceedings First Workshop on *CTP Components for Educational Software,* Wrocław, Poland, 31th July 2011, volume 79 of *Electronic Proceedings in Theoretical Computer Science*, pages 63–81. Open Publishing Association, 2012.
13. Vesna Marinković. ArgoTriCS – Automated Triangle Construction Solver. *Journal of Experimental and Theoretical Artificial Intelligence*, 2015. To appear.

14. Vesna Marinković. On-line compendium of triangle construction problems with automatically generated solutions. *The Teaching of Mathematics*, pages 29–44, 2015.
15. Vesna Marinkovic, Predrag Janicic, and Pascal Schreck. Computer theorem proving for verifiable solving of geometric construction problems. In Francisco Botana and Pedro Quaresma, editors, *Automated Deduction in Geometry - 10th International Workshop, ADG 2014, Coimbra, Portugal, July 9-11, 2014, Revised Selected Papers*, volume 9201 of *Lecture Notes in Computer Science*, pages 72–93. Springer, 2014.
16. Vesna Marinković and Predrag Janičić. Towards Understanding Triangle Construction Problems. In J. Jeuring et al., editor, *Intelligent Computer Mathematics - CICM 2012*, volume 7362 of *Lecture Notes in Computer Science*. Springer, 2012.
17. Vesna Marinković, Predrag Janičić, and Pascal Schreck. Solving Geometric Construction Problems Supported by Theorem Proving. In *Proceedings of the 10th International Workshop on Automated Deduction in Geometry (ADG 2014)*, pages 121–146. CISUC Technical Report TR 2014/01, University of Coimbra, 2014.
18. Leroy F Meyers. Update on William Wernick's Triangle Constructions with Three Located Points. *Mathematics Magazine*, 69(1):46–49, 1996.
19. Lawrence C. Paulson. *Isabelle: a Generic Theorem Prover*, volume 828 of *Lecture Notes in Computer Science*. Springer, 1994.
20. Pascal Schreck and Pascal Mathis. RC-Constructibility of Problems in Wernick's List. In *Proceedings of the 10th International Workshop on Automated Deduction in Geometry (ADG 2014)*, pages 85–104. CISUC Technical Report TR 2014/01, University of Coimbra, 2014. available at URL `http://www.uc.pt/en/congressos/adg/adg2014/program/proceedings`.
21. Eckehard Specht. Wernicks Liste (in Deutsch). `http://hydra.nat.uni-magdeburg.de/wernick/`.
22. Alexey V. Ustinov. On the Construction of a Triangle from the Feet of Its Angle Bisectors. *Forum Geometricorum*, 9:279–280, 2009.
23. William Wernick. Triangle constructions vith three located points. *Mathematics Magazine*, 55(4):227–230, 1982.
24. Paul Yiu. Elegant Geometric Constructions. *Forum Geometricorum*, 5:7596, 2005.
25. Stefanie Zukowski. Approaching Constructability of Triangles. In *Young Mathematicians Conference*. The Ohio State University, 2010.

Appendix

Summary of our results

We recall here the results used in [20], by giving coordinates of the characteristic points of the problem, and then the last equation of the systems obtained after triangularization using the Maple implementation of regular chains. Fortunately, testing the last equation was enough for the open problems.

Wernick 77 : O, H_a, T_b
Coordinates: $(0,0)$, $(-1,-3)$ and $(-3,0)$.

$$84349y_A^8 + 668100y_A^7 + 908434y_A^6 - 6940782y_A^5 - 32743501y_A^4 - 63643476y_A^3 - 72253168y_A^2 - 56499066y_A - 25568010$$

the splitting field of which has degree $8! = 40320$ which is not a power of 2: this problem is not RC-constructible

Wernick 78 : I, O, H_a
Coordinates: $(0,0)$, $(0,1)$,$(-1,-3)$.

$P(y_C) = 325y_C^8 + 2050y_C^7 - 75y_C^6 - 11256y_C^5 + 7749y_C^4 + 8964y_C^3 - 107730y_C^2$
$+ 160380y_C - 14580$

the splitting field of which has degree $\frac{8!}{105} = 384$ which is not a power of 2. Therefore, this problem is not RC-constructible.

Wernick 81: O, T_a, T_b
Coordinates: $(0,0)$, $(-1,-3)$ and $(-3,0)$.

$P(y_C) = 5202928809y_C^8 + 34323168906y_C^7 + 64988457138y_C^6 - 168831818766y_C^5 - 1131189431845y_C^4 - 2336530456944y_C^3 - 2257027274736y_C^2 - 1030105859328y_C - 178376649984$

the splitting field of which has degree $8! = 40320$. Therefore, this problem is not RC-constructible.

Wernick 113: T_c, T_b, M_a
Coordinates: $(0,0)$, $(2,2)$ and $(4,0)$. We get two systems, for the first one we have the polynomial:

$P(y_C) = 25y_C^3 - 94y_C^2 + 160y_C - 128$

and for the second one:

$P(y_C) = 3y_C^3 - 10y_C^2 + 60y_C - 72$

Therefore, this problem is not RC-constructible.

Wernick 118: T_b, H_a, G
Coordinates $T_b(0,0)$, $H_a(6,0)$ and $G(4,3)$

$P(y_C) = y_C^5 + 136y_C^4 - 848y_C^3 + 14112y_C^2 - 52164y_C + 52488$

Therefore, this problem is not RC-constructible.

Wernick 119 I, H_a, G
When choosing coordinates $(0,0)$ for I, $(1,-2)$ for H_a and $(1,1)$ for G, we obtain two systems. The second one corresponds to non real solutions, and the first one

contains the following polynomial of degree 4:

$$P(y_C) = 289y_C^4 - 867y_C^3 - 57528y_C^2 - 99144y_C - 41472$$

the splitting field of which has degree 8 over $\mathbb{Q}$. This result does not mean that the problem is RC-constructible. In order to prove its RC-constructibility, we have to take parameters a and b as coordinates for one of the three points and then compute its Galois group. The triangularization produces a huge system displayed with more than 400 lines and the coefficient of the degree 4 term of the irreducible polynomial we want to test is :

$$19683a^9 - 59049a^8 + (78732b^2 + 61236)a^7 + (-183708b^2 - 20412)a^6 + (118098b^4 + 166212b^2 - 4374)a^5 + (-196830b^4 - 72900b^2 + 2754)a^4 + (78732b^6 + 148716b^4 + 10692b^2 + 324)a^3 + (-78732b^6 - 61236b^4 + 3564b^2 - 108)a^2 + (19683b^8 + 43740b^6 + 18954b^4 - 756b^2 - 21)a - 6561b^8 - 8748b^6 - 3078b^4 - 108b^2 - 1$$

Maple is powerful enough to compute Galois' group of this huge parameterized polynomial and find:

$$"4T3", \{"D(4)"\}, " - ", 8, \{"(13)", "(1234)"\}$$

From this result, we can conclude that the problem is RC-constructible.

We confirm that result by using Gao and Chou's method [8]. This method leads to heavy computations but allows, in principle, to extract a RC-construction. Unfortunately, it is almost impossible for this concrete problem. The equation of degree 3 considered in that method is huge: this is, for the sake of illustration, just the coefficients for the term of degree 3:

$$12754584a^{13} + 76527504a^{11}b^2 + 191318760a^9b^4 + 255091680a^7b^6 + 191318760a^5b^8 + 76527504a^3b^{10} + 12754584ab^{12} - 55269864a^{12} - 280600848a^{10}b^2 - 573956280a^8b^4 - 595213920a^6b^6 - 318864600a^4b^8 - 76527504a^2b^{10} - 4251528b^{12} + 93533616a^{11} + 416649744a^9b^2 + 731262816a^7b^4 + 629226144a^5b^6 + 263594736a^3b^8 + 42515280ab^{10} - 72748368a^{10} - 314613072a^8b^2 - 515852064a^6b^4 - 387361440a^4b^6 - 121877136a^2b^8 - 8503056b^{10} + 18108360a^9 + 115263648a^7b^2 + 214465968a^5b^4 + 155574432a^3b^6 + 38263752ab^8 + 8030664a^8 - 4408992a^6b^2 - 48813840a^4b^4 - 42200352a^2b^6 - 5826168b^8 - 4269024a^7 - 11547360a^5b^2 + 1469664a^3b^4 + 9867744ab^6 - 536544a^6 + 2309472a^4b^2 + 2869344a^2b^4 - 1469664b^6 + 355752a^5 + 618192a^3b^2 - 390744ab^4 + 55080a^4 - 89424a^2b^2 - 71928b^4 - 9936a^3 - 24624ab^2 - 3024a^2 - 1296b^2 - 264a - 8$$

Wernick 122: T_b, T_a, G

With coordinates $T_b(0,0)$, $T_a(4,0)$ and $G(2,1)$, we get two systems containing the disqualifying equations:

$$P(y_C) = y_C^4 - 6y_C^3 - 51y_C^2 - 24y_C + 36$$

the splitting field of which is of order 24 and

$P(y_C) = 2701y_C^3 - 12871y_C^2 + 43008y_C - 28224$

Therefore, this problem is not RC-constructible.

Wernick 123: I**,** T_a**,** G
With the coordinates $(0,0)$, $(4,0)$ and $(2,1)$, we obtain three irreducible triangular systems, but the last one does not have real solutions. The first one contains the polynomial:

$P(y_C) = 98596y_C^8 - 533172y_C^7 + 1934365y_C^6 - 2612838y_C^5 + 541114y_C^4 + 2325666y_C^3 + 162729y_C^2 - 3815532y_C + 1555848$

the Galois group of which is:
$"8T44", "[2^4]S(4)", " - ", 384, "(48)", "(18)(45)", "(1238)(4567)"$

And the second one

$P(y_C) = 4y_C^6 - 36y_C^5 + 192y_C^4 - 612y_C^3 + 81y_C^2 + 2025y_C - 3402$

Therefore, this problem is not RC-constructible.

Wernick 127: T_c**,** H_b**,** H_a
Coordinates $(0,0)$, $(0,-6)$ and $(6,-2)$.

$P(y_C) = 8125y_C^4 + 146484y_C^3 + 830844y_C^2 + 1715040y_C + 1049760$

The splitting field of which has degree 24 over $\mathbb{Q}$. Therefore, this problem is not RC-constructible.

Wernick 128: T_c**,** H_b**,** H_a
Coordinates $(0,0)$, $(0,-6)$ and $(6,-2)$.

$P(y_C) = 8125y_C^4 + 146484y_C^3 + 830844y_C^2 + 1715040y_C + 1049760$

which is not RC-resolvable since its splitting field has degree 24. Therefore, this problem is not RC-constructible.

Wernick 132: T_a**,** T_b**,** H_a
Coordinates $(0,0)$, $(4,0)$, and $(-1,3)$.

$P(y_C) =$
$9825y_C^6 - 72620y_C^5 + 691848y_C^4 - 403200y_C^3 + 1442880y_C^2 + 10886400y_C - 15552000$

Therefore, this problem is not RC-constructible.

Wernick 134: T_c**,** T_b**,** H_a
Coordinates $(0,0)$, $(0,2)$ and $(2,1)$.

$P(y_C) =$
$524475y_C^8 - 5345280y_C^7 + 24048076y_C^6 - 62358704y_C^5 + 102412544y_C^4 - 109631360y_C^3 + 75046720y_C^2 - 30134400y_C + 5432000$

the Galois group of which is of order 384. Therefore, this problem is not RC-constructible.

Wernick 135: I**,** T_b**,** H_a
With points $I(0,0)$, $T_b(0,2)$ and $H_a(2,-1)$, we get two systems. In the first one, we have the polynomial:

$P(y_C) =$
$58968y_C^8 - 194436y_C^7 + 453056y_C^6 - 311496y_C^5 + 319980y_C^4 - 526960y_C^3 + 466030y_C^2 - 210025y_C + 28000$

the splitting field of which has degree 40320. And we have in the second one:

$P(y_C) = 572y_C^5 - 1624y_C^4 + 2088y_C^3 + 2532y_C^2 - 585y_C + 1200$

Therefore, this problem is not RC-constructible.

Wernick 136: T_a**,** T_b**,** H
With points $T_a(0,0)$, $T_b(4,0)$ and $H(2,-1)$, we get two systems. The first one contains the polynomial:

$P(y_C) = 15y_C^4 - 8y_C^3 - 148y_C^2 - 32y_C + 192$

the splitting field of which has degree 24. And we have in the second one:

$P(y_C) = 5705y_C^3 + 25412y_C^2 + 12288y_C - 9216$

Therefore, this problem is not RC-constructible.

Wernick 137 : I, T_a, H
Coordinates $(0,0)$, (a,b) and $(0,-2)$. We take parameters as coordinates of T_a as we thought that the problem was constructible. We obtain two systems after more than 6 hours of computation. The following polynomial in y_A is the last equation of the first component

$(9a^4 + (18b^2 + 36b - 12)a^2 + 9b^4 + 36b^3 + 84b^2 + 96b + 64)y_A^4$
$+ (18a^4 + (78b^2 + 192b + 48)a^2 + 60b^4 + 192b^3 + 352b^2 + 288b + 128)y_A^3$
$+ ((30b+36)a^4 + (30b^3 + 160b^2 + 256b + 96)a^2 + 148b^4 + 384b^3 + 528b^2 + 320b + 64)y_A^2$

$$+((24b+24)a^4+(96b^3+224b^2+160b+32)a^2+160b^4+352b^3+320b^2+128b)y_A$$
$$+(24b^2+24b)a^4+(80b^3+112b^2+32b)a^2+64b^4+128b^3+64b^2$$

the Galois group of which is:
$"4T5", "S(4)", "-", 24, "(14)", "(24)", "(34)"$

The second system provides an equation of degree 3. We can then conclude that this problem is not RC-constructible.