Visit to University of Rome and Presentation of the GCLC Tool

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ARGO Seminar
Faculty of Mathematics, University of Belgrade
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Visit to Rome

- Department of Mathematics, University Roma 1 ("La Sapienza")

- Two talks on GCLC given

- Several meetings held

Agenda

- What is dynamic geometry software?

- Dynamic geometry software GCLC

- Coordinate-free and coordinate-based methods for automated theorem proving in geometry:

- Theorem provers built-into GCLC

- Intelligent mathematical software

What is Dynamic Geometry Software?

- Interactive geometry software or Dynamic geometry software or Dynamic geometry environments or Dynamic geometry tools

- DG tools allow the user "to create and then manipulate geometric constructions, primarily in plane geometry"

- The user typically starts a construction with a few points, construct new objects, and then can move the points to see how the construction changes

What Good is Dynamic Geometry Software?

- Good for students

- Good for teachers

- Good for publishing

Some Dynamic Geometry Tools

- Cabri Geometry — since 1988

- Geometer Sketchpad (GSP) — since 1991

- Cinderella (different geometries)

- KSEG, Eukleides, DrGeo

- 3D tools: Cabri 3D, Archimedes Geo3D, JavaView

- http://en.wikipedia.org/wiki/Dynamic_geometry_software

Different Tools, Different Skills

- Animations, loci, ...

- Symbolic expressions, calculations, ...

- Saving constructions, saving figures, ...

- Multilingual

- Automated theorem proving, probabilistic proofs, ...

GCLC/WinGCLC

- First version released in 1996, originally, as a tool for producing geometrical illustrations for LaTeX, hence the name GCLC:

  "Geometry Constructions → LaTeX Converter"

- Command-line versions for Windows and Linux and a version with graphical interface for Windows (WinGCLC)

- Freely available from `http://www.matf.bg.ac.yu/~janicic/gclc` and from EMIS (The European Mathematical Information Service) servers `http://www.emis.de/misc/index.html`

Main applications of GCLC/WinGCLC

- producing digital mathematical illustrations

- mathematical education

- storing mathematical contents

- studies of automated geometrical reasoning

GCLC Users

- Thousands users worldwide — used in a number of high-schools and university courses, and for publishing

- >23000 visitors since 2003, last 2000 visitors:

GCLC: Basic Principles

- A construction is a formal procedure, not an image

- Producing mathematical illustrations should be based on "describing figures", not on "drawing figures" (similarly as T_EX)

- Images can be produced from descriptions, but not vice-versa!

- All instructions are given explicitly, in GCLC language

- GCLC language is like a simple programming language, easily understandable to mathematicians

Features (part I)

- Support for geometrical primitive constructions, compound constructions, transformations, etc.

- Symbolic expressions, while-loops, user-defined procedures

- Conics, 2D and 3D curves, 3D surfaces

- Log files with information on all objects

- Built-in theorem provers

Features (part II)

- User-friendly interface, interactive work, animations, traces

- Export to different formats (LaTeX— several versions, EPS, BMP, SVG), import from JavaView

- Full XML support

- Free, small in size (<1Mb), easy to use, well documented

GCLC Language

- Instructions for describing **contents**

- Instructions for describing **presentation**

- All of them are explicit, given by GCLC commands

## Simple Example (part I)

```
% fixed points                    |    % labelling points
point A 15 20                      |    cmark_lb A
point B 80 10                      |    cmark_rb B
point C 70 90                      |    cmark_rt C
                                   |    cmark_lt O_1
% side bisectors                   |    cmark_rt O_2
med a B C                          |
med b A C                          |    % drawing the sides of the triangle ABC
med c B A                          |    drawsegment A B
                                   |    drawsegment A C
% intersections of bisectors       |    drawsegment B C
intersection O_1 a b               |
intersection O_2 a c               |    % drawing the circumcircle of the triangle
                                   |    drawcircle O_1 A
```
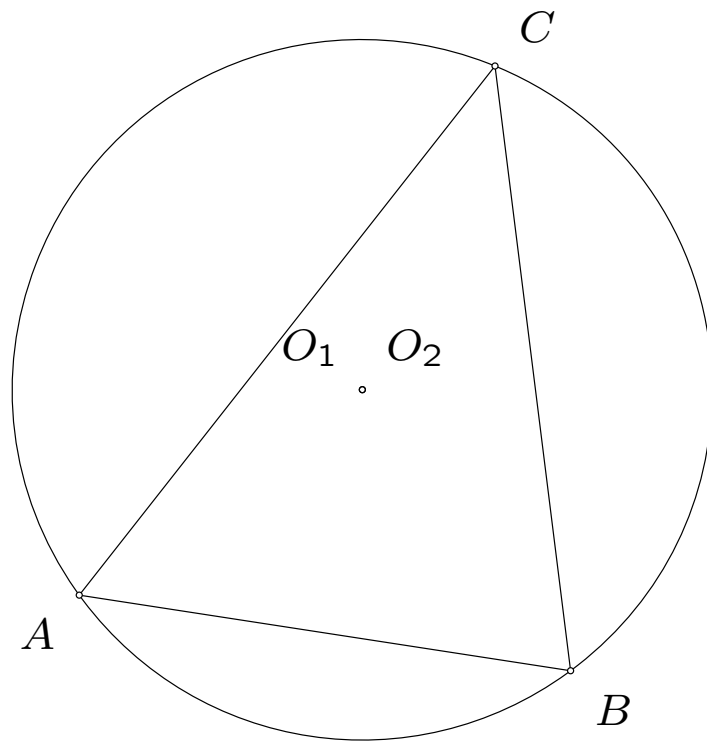
Overview of the Language

- Basic definitions, constructions, transformations

- Drawing, labelling, and printing commands

- 2D and 3D Cartesian commands

- Symbolic expressions, loops, user-defined procedures

- Commands for describing animations

- Commands for the geometry theorem proving

Demo: Several GCLC Examples

Methods for Automated Theorem Proving in Geometry

- Coordinate-free methods - give traditional (human readable) proofs:

    - Gelertner's theorem prover (Gelertner 1950's)

    - Area, Angle method (Chou et.al. 1990's)

- Algebraic methods (no synthetic geometry proofs, just algebraic arguments):

    - Gröbner basis method (Buchberger 1965)

    - Wu's method (Wu 1977)

Geometrical Theorems of Constructive Type

- Conjectures that corresponds to properties of constructions

- Usually, only Euclidean plane geometry

- Non-degenerate conditions are very important

Area method

The method deals with the following geometry quantities:

ratio of directed segments: for four collinear points P, Q, A, and B such that $A \neq B$, it is the ratio $\dfrac{\overrightarrow{PQ}}{\overrightarrow{AB}}$;

signed area: it is the signed area S_{ABC} of a triangle ABC or the signed area S_{ABCD} of a quadrilateral $ABCD$;

Area method (2)

Pythagoras difference: for three points, P_{ABC} is defined as fol-
lows:

$$P_{ABC} = AB^2 + CB^2 - AC^2 \ .$$

Pythagoras difference for four points, P_{ABCD} is defined as
follows:

$$P_{ABCD} = P_{ABD} - P_{CBD} \ .$$

real number: it is a real number, constant.

Area method (3)

- All construction steps are reduced to a limited number of specific constructions

- The conjecture is also expressed as an equality over geometry quantities (over points already introduced)

- The goal is to prove the conjecture by reducing it to a trivial equality (0=0)

Area method (4)

points A and B are identical	$P_{ABA} = 0$
points A, B, C are collinear	$S_{ABC} = 0$
AB is perpendicular to CD	$P_{ACD} = P_{BCD}$
AB is parallel to CD	$S_{ACD} = S_{BCD}$
O is the midpoint of AB	$\dfrac{\overrightarrow{AO}}{\overrightarrow{OB}} = 1$
AB has the same length as CD	$P_{ABA} = P_{CDC}$
points A, B, C, D are harmonic	$\dfrac{\overrightarrow{AC}}{\overrightarrow{CB}} = \dfrac{\overrightarrow{DA}}{\overrightarrow{DB}}$

Area method (5)

- For reducing the goal, different simplifications are used:

$$\begin{aligned}
x \cdot 1 &\rightarrow x \\
x \cdot 0 &\rightarrow 0 \\
S_{AAB} &\rightarrow 0 \\
S_{ABC} &\rightarrow S_{BCA}
\end{aligned}$$

- Crucially, for each pair quantity-construction step there is one *elimination lemma* that enable eliminating a relevant point

- Thank to these lemmas, the point are eliminated from the conjecture in opposite direction that they were introduced one by one

Area Method — Elimination lemmas

For instance, if a point Y was introduced as the intersection of lines UV and PQ, then Y can be eliminated from expression of the form $\dfrac{\overrightarrow{AY}}{\overrightarrow{CD}}$ using the following equality:

$$\frac{\overrightarrow{AY}}{\overrightarrow{CD}} = \begin{cases} \dfrac{S_{APQ}}{S_{CPDQ}} \, , & \text{if } A \in UV \\[2ex] \dfrac{S_{AUV}}{S_{CUDV}} \, , & \text{if } A \notin UV \end{cases}$$

Example: Menelaus's Theorem
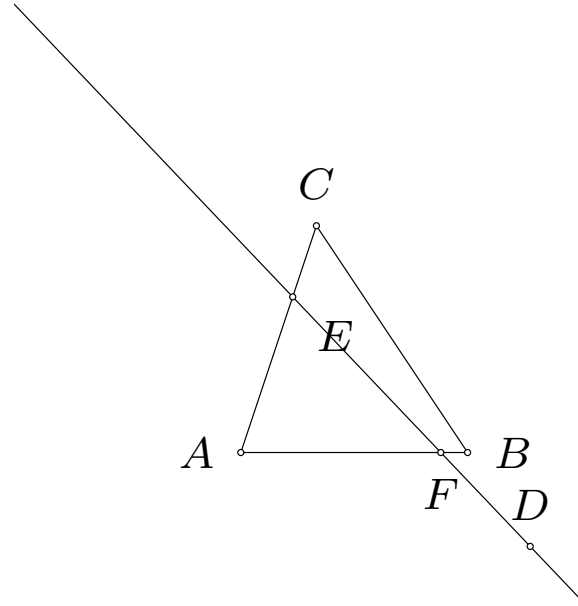
- Conjecture:

$$\frac{\overrightarrow{AF}}{\overrightarrow{FB}} \cdot \frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}} = -1$$

Example: Menelaus's Theorem (2)

- Fragment of the proof:

$$\left(\frac{\overrightarrow{AF}}{\overrightarrow{BF}} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right)\right) = 1, \text{ by algebraic simplifications}$$

$$\left(\frac{S_{ADE}}{S_{BDE}} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right)\right) = 1, \text{ by Lemma 8 (point } F \text{ eliminated)}$$

...

$0 = 0$, by algebraic simplifications

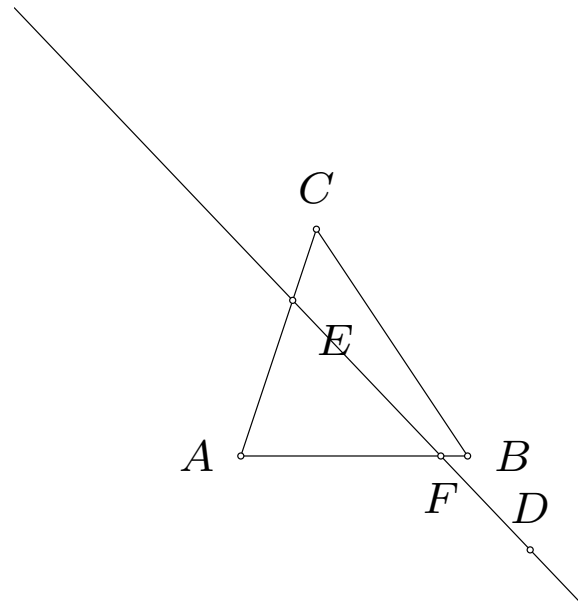# Coordinate-based (Algebraic) methods

- Geometry statements have the form of equalities

- Construction steps are converted into a polynomial system

$$
\begin{aligned}
h_1(u_1, u_2, \ldots, u_d, x_1, \ldots, x_n) &= 0 \\
h_2(u_1, u_2, \ldots, u_d, x_1, \ldots, x_n) &= 0 \\
&\cdots \\
h_t(u_1, u_2, \ldots, u_d, x_1, \ldots, x_n) &= 0
\end{aligned}
$$

- The goal is to check whether for the conjecture it holds that

$$
g(u_1, u_2, \ldots, u_d, x_1, \ldots, x_n) = 0
$$

Example: Menelaus Theorem

- Coordinates assigned to the points:

$$A(0,0), B(u_1,0), C(u_2,u_3), D(x_1,u_4), E(x_2,u_5), F(x_4,0)$$

Example: Menelaus Theorem (2)

- Conditions:

  D on BC: $p_1 = -u_3x_1 + (u_4u_2 - u_4u_1 + u_3u_1)$

  E on AC: $p_2 = -u_3x_2 + u_5u_2$

  F on DE: $p_3 = (-u_5 + u_4)x_4 - u_4x_2 + u_5x_1$

- Conjecture:

$$p_4 = (-u_5u_3 + u_4u_3)x_4 + (-u_5u_4u_1 + u_5u_3u_1)$$

Wu's Method

- Invented by Wu in 1977

- Considered to be the most efficient method for automated theorem proving in all fields (not only geometry)

- Considered to be one of the four modern great Chinese inventions

- Similar to Gauss' elimination procedure

Wu's Method on Menelaus Theorem

- For the above example, triangulation gives:

$$
\begin{aligned}
p_1 &= -u_3 x_1 + (u_4 u_2 - u_4 u_1 + u_3 u_1) \\
p_2 &= -u_3 x_2 + u_5 u_2 \\
p_3 &= (-u_5 + u_4) x_4 - u_4 x_2 + u_5 x_1
\end{aligned}
$$

- Wu's elimination procedure in several steps gives $p_4 = 0$, which was required to prove

Gröbner-bases Method

- Invented by Buchberger in 1965, widely used CAS algorithm with many applications

- Gröbner basis (GB) is a particular kind of generating subset of an ideal of a polynomial ring R.

- Buchberger's algorithm builds GB for the set of polynomials corresponding to the construction and then it checks the conjecture, by efficiently testing whether its remainder with respect to GB is 0

- For reducing w.r.t. the Gröbner base, the ordering of reducing is irrelevant

Theorem Provers Built-into GCLC

- There are three theorem provers built-into GCLC:

    – a theorem prover based on the area method

    – a theorem prover based on the Wu's method

    – a theorem prover based on the Buchberger's method

- All of them are very efficient and can prove many non-trivial theorems in only milliseconds.

Using Theorem Provers Built-into GCLC

- The theorem provers are tightly built-in: the user has just to state the conjecture about the construction described.

- For example:

```
prove { identical O_1 O_2 }
```

Demo: Several Examples

- The repository GeoThms `http://hilbert.mat.uc.pt/~geothms` (developed by Pedro Quaresma (Portugal) and Predrag Janičić) contains >100 theorems automatically proved

- Most of these theorems are included in the GCLC distribution available from the Internet

Processing Descriptions of Constructions

- Syntactical check

- Semantical check (e.g., whether two concrete points deter-
  mine a line)

- Deductive check — verifies if a construction is regular (e.g., whether
  two constructed points never determine a line)

Intelligent Geometrical Software

Intelligent Geometry Software
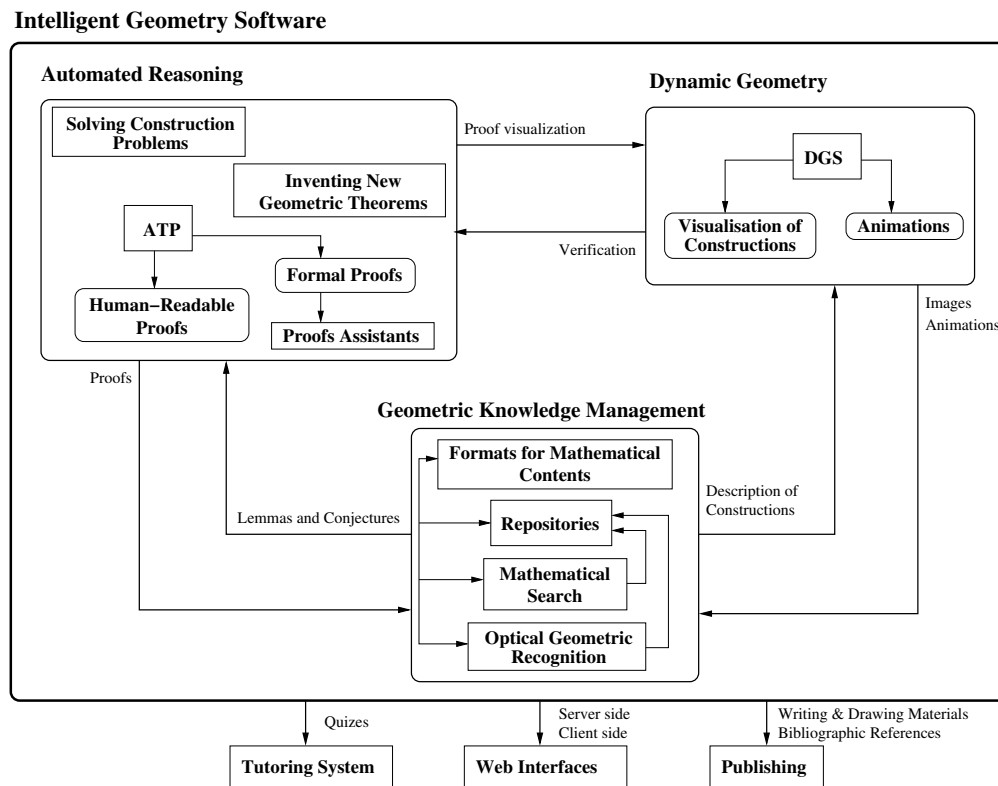
Conclusions

- Dynamic geometry tools are around for twenty years but just recently they started to be really intelligent

- Automated geometrical theorem provers are around for forty years but just recently they started to work in harmony with dynamic geometry tools

- GCLC aims to be a powerful and intelligent geometrical assistant