GeoCoq: a library for foundations of geometry

Pierre Boutry

IGG

Belgrade, June 12, 2025

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- The **missing** concept in *Euclid's Elements*.

Euclid
(325 B.C. - 265 B.C.)

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Moritz Pasch
(1843 - 1930)

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Adrien-Marie Legendre
(1752 - 1833)

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Vladimir Voevodsky
(1966 - 2017)

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  It soon became clear that the only real long-term solution to the problems that I encountered is to start using computers in the verification of mathematical reasoning.

(Vladimir Voevodsky, talk in March 2014 at the Institute for Advanced Studies at Princeton)

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Proof assistants

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- 11 axioms.
- A parameter controls the dimension.
- Good meta-theoritical properties.

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Axiom (Identity for congruence)

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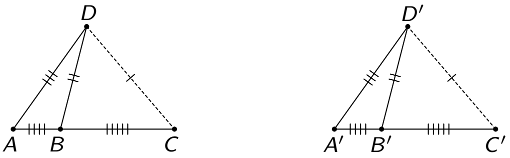

Axiom of Segment Construction

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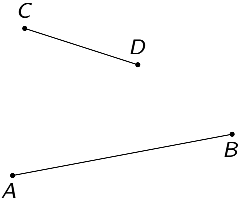
Axiom (Segment Construction)

$$\exists E, A\text{—}B\text{—}E \land BE \equiv CD$$



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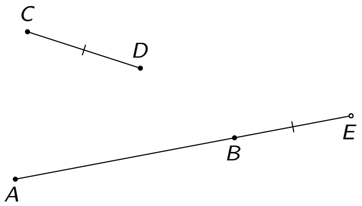

Pasch's axiom

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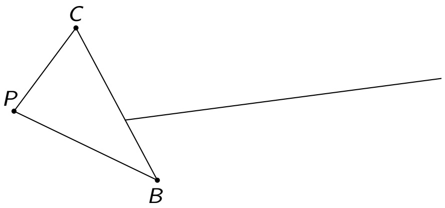

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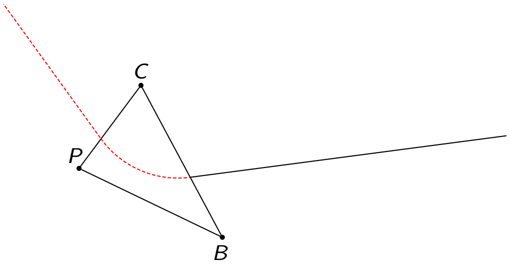

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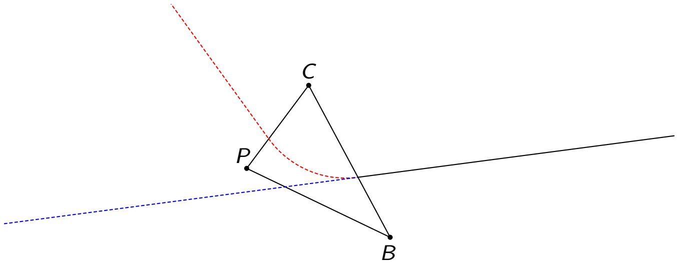

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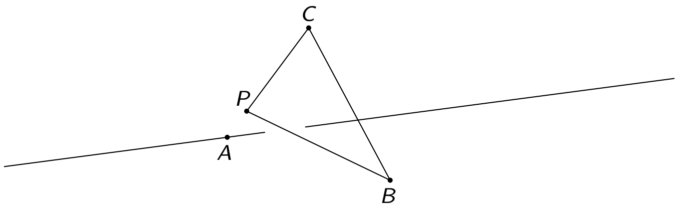

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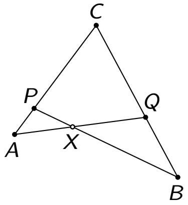

2-Dimensional Axiom

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Axiom (Lower 2-Dimensional)

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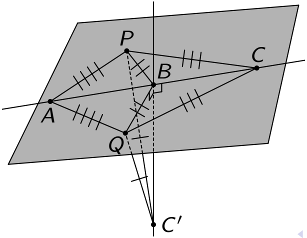

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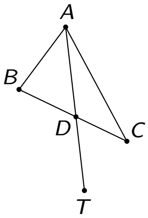

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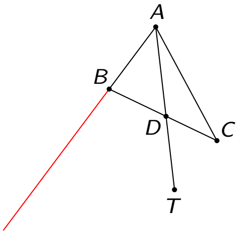

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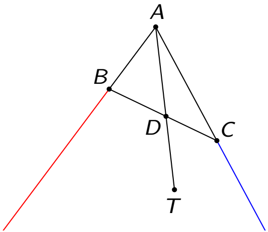

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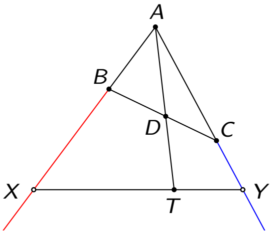

The axioms

Identity for betweenness	$A—B—A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
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Segment Construction	$\exists E, A—B—E \wedge BE \equiv CD$
Pasch	$A—P—C \wedge B—Q—C \Rightarrow \exists X, P—X—B \wedge Q—X—A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A—B—C \wedge A'—B'—C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
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Point equality decidability	$X = Y \vee X \neq Y$

Overview of the formalization

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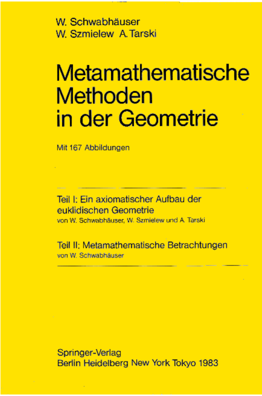

W. Schwabhäuser
W. Szmielew A. Tarski

Metamathematische Methoden in der Geometrie

Mit 167 Abbildungen

Teil I: Ein axiomatischer Aufbau der euklidischen Geometrie
von W. Schwabhäuser, W. Szmielew und A. Tarski

Teil II: Metamathematische Betrachtungen
von W. Schwabhäuser

Springer-Verlag
Berlin Heidelberg New York Tokyo 1983

geocoq.github.io/GeoCoq/

An example of proof by computation

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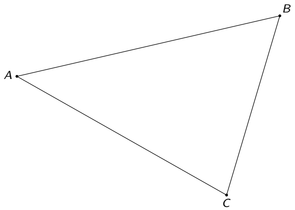

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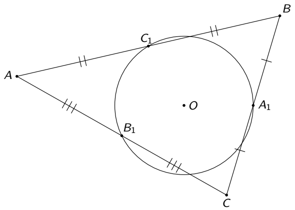

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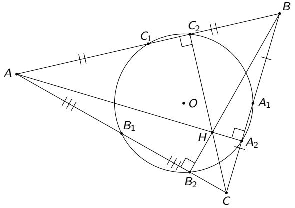

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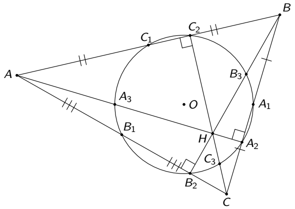

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- This establishes the *relative* consistency of the theory.

An overview of the GeoCoq library

Questions in foundations of geometry

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- Study of axiom systems.

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Questions in foundations of geometry

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- Study of some axioms: their classification.
- Properties of axiom systems: is there some proof of false? can we simplify the system? ...

Ways to axiomatize Euclidean geometry

Ways to axiomatize Euclidean geometry

- Synthetic approach

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Euclid
(325 BC - 265 BC)

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(1862 - 1943)

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Ways to axiomatize Euclidean geometry

- Synthetic approach: geometric objects and axioms about them.
  - Euclid
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  - Tarski
- Analytic approach: a field $\mathbb{F}$ is assumed and the space is defined as $\mathbb{F}^n$.
- Mixed analytic/synthetic approach: existence of a field and geometric axioms.
  - Birkhoff
- Erlangen program: a geometry is defined as a space of objects and a group of transformations acting on it.

Arithmetization of geometry

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René Descartes
(1596 - 1650)

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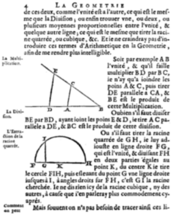

4 LA GEOMETRIE

de ces deux, comme l'vnité est a l'autre, ce qui est le mesme que la Diuision ; ou enfin trouuer vne, ou deux, ou plusieurs moyennes proportionnelles entre l'vnité, & quelque autre ligne ; ce qui est le mesme que tirer la racine quarrée, ou cubique, &c. Et ie ne craindray pas d'introduire ces termes d'Arithmetique en la Geometrie, afin de me rendre plus intelligible.

La Multiplication.

Soit par exemple A B l'vnité, & qu'il faille multiplier B D par B C, ie n'ay qu'a ioindre les poins A & C, puis tirer DE parallele a CA, & BE est le produit de cette Multiplication.

La Diuision.

Oubien s'il faut diuiser BE par BD, ayant ioint les poins E & D, ie tire A C parallele a DE, & BC est le produit de cette diuision.

L'Extraction de la racine quarrée.

Ou s'il faut tirer la racine quarrée de G H, ie luy adiouste en ligne droite F G, qui est l'vnité, & diuisant FH en deux parties égales au point K, du centre K ie tire le cercle FIH, puis esleuant du point G vne ligne droite iusques à I, à angles droits sur F H, c'est G I la racine cherchée. Ie ne dis rien icy de la racine cubique, ny des autres, à cause que i'en parleray plus commodement cy-aprés.

Comment on peut

Mais souuent on n'a pas besoin de tracer ainsi ces li-

A page from *La Géométrie* of Descartes

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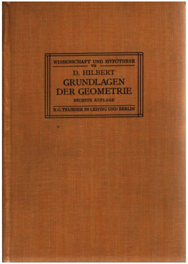

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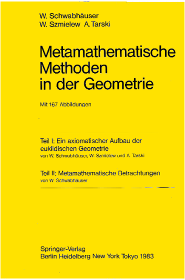

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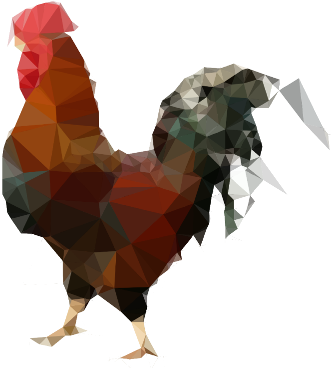
- Not all versions of the parallels postulate allow for the arithmatization of geometry in a constructive setting.

Arithmetization of geometry



geocoq.github.io/GeoCoq/

Arithmetization of geometry

- The only library to have formalized the arithmetization of geometry.

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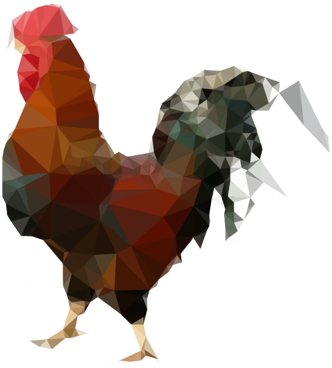

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Types of independence proofs

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- Semantic proofs: prove the consistency of non-Euclidean geometry.

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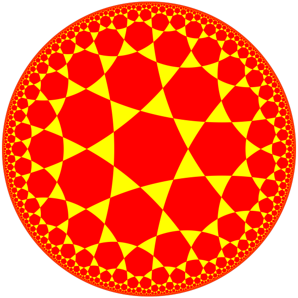

Poincaré disk

Types of independence proofs

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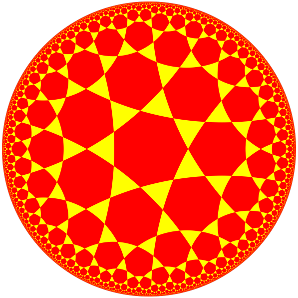

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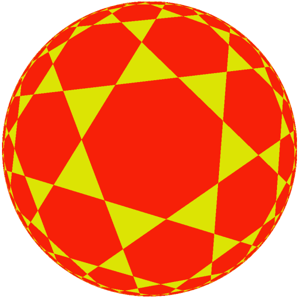

Klein model

Types of independence proofs

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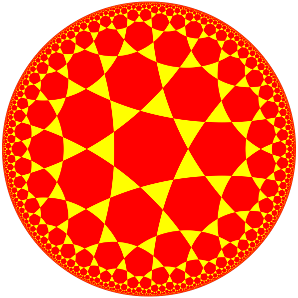

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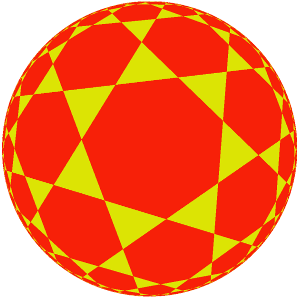

Klein model

- Syntactic proofs: prove there does not exist a derivation of the axiom from the others.

Syntactic proof

Identity for betweenness	$A—B—A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A—B—E \wedge BE \equiv CD$
Pasch	$A—P—C \wedge B—Q—C \Rightarrow \exists X, P—X—B \wedge Q—X—A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A—B—C \wedge A'—B'—C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A—B—C \wedge \neg B—C—A \wedge \neg C—A—B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A—B—C \vee B—C—A \vee C—A—B$
Euclid	$A—D—T \wedge B—D—C \wedge A \neq D \Rightarrow$ $\exists XY, A—B—X \wedge A—C—Y \wedge X—T—Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, X \in \Xi \wedge Y \in \Upsilon \Rightarrow A—X—Y)) \Rightarrow$ $\exists B, (\forall XY, X \in \Xi \wedge Y \in \Upsilon \Rightarrow X—B—Y)$
Point equality decidability	$X = Y \vee X \neq Y$

Syntactic proof

Identity for betweenness	$A\text{—}B\text{—}A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \land AB \equiv EF \Rightarrow CD \equiv EF$
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Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \land BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \land B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \land Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \land BC \equiv B'C' \land$ $AD \equiv A'D' \land BD \equiv B'D' \land$ $A\text{—}B\text{—}C \land A'\text{—}B'\text{—}C' \land A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \land \neg B\text{—}C\text{—}A \land \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \Rightarrow$ $A\text{—}B\text{—}C \lor B\text{—}C\text{—}A \lor C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \land B\text{—}D\text{—}C \land A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \land A\text{—}C\text{—}Y \land X\text{—}T\text{—}Y$

Syntactic proof

Identity for betweenness	$A–B–A \Rightarrow A = B$
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Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A–B–C \wedge A'–B'–C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A–B–C \wedge \neg B–C–A \wedge \neg C–A–B$
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Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
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Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$

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Euclid	$A{-}D{-}T \wedge B{-}D{-}C \wedge A \neq D \Rightarrow$ $\exists XY, A{-}B{-}X \wedge A{-}C{-}Y \wedge X{-}T{-}Y$

Syntactic proof

$A\text{—}B\text{—}A \Rightarrow A = B$

$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$

$AB \equiv BA$

$AB \equiv CC \Rightarrow A = B$

$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$

$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$

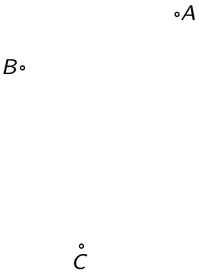
$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$
$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$
$A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$

$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$

$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$
$A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$

$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$
$\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$

Syntactic proof



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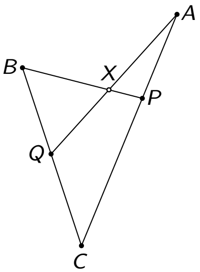

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$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$

$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$

$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$
$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$
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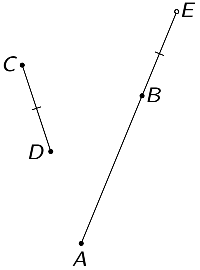

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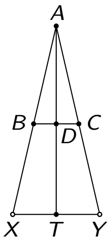

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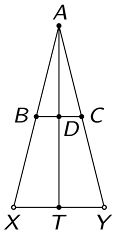
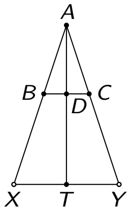
$AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D' \wedge A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow \exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$

Syntactic proof



$A{-}B{-}A \Rightarrow A = B$
$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
$AB \equiv BA$
$AB \equiv CC \Rightarrow A = B$
$\exists E, A{-}B{-}E \wedge BE \equiv CD$
$A{-}P{-}C \wedge B{-}Q{-}C \Rightarrow \exists X, P{-}X{-}B \wedge Q{-}X{-}A$
$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A{-}B{-}C \wedge A'{-}B'{-}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
$\exists ABC, \neg A{-}B{-}C \wedge \neg B{-}C{-}A \wedge \neg C{-}A{-}B$
$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A{-}B{-}C \vee B{-}C{-}A \vee C{-}A{-}B$
$A{-}D{-}T \wedge B{-}D{-}C \wedge A \neq D \Rightarrow$ $\exists XY, A{-}B{-}X \wedge A{-}C{-}Y \wedge X{-}T{-}Y$

Syntactic proof



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$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
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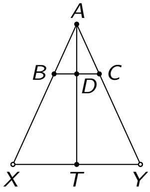
$\exists E, A - B - E \wedge BE \equiv CD$
$A - P - C \wedge B - Q - C \Rightarrow \exists X, P - X - B \wedge Q - X - A$
$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A - B - C \wedge A' - B' - C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
$\exists ABC, \neg A - B - C \wedge \neg B - C - A \wedge \neg C - A - B$
$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A - B - C \vee B - C - A \vee C - A - B$
$A - D - T \wedge B - D - C \wedge A \neq D \Rightarrow$ $\exists XY, A - B - X \wedge A - C - Y \wedge X - T - Y$

Syntactic proof



$A\text{—}B\text{—}A \Rightarrow A = B$
$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
$AB \equiv BA$
$AB \equiv CC \Rightarrow A = B$
$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
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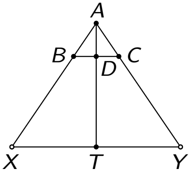
Syntactic proof



A—B—$A \Rightarrow A = B$

$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$

$AB \equiv BA$

$AB \equiv CC \Rightarrow A = B$

$\exists E, A$—B—$E \wedge BE \equiv CD$

A—P—$C \wedge B$—Q—$C \Rightarrow \exists X, P$—$X$—$B \wedge Q$—$X$—$A$

$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$
$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$
A—B—$C \wedge A'$—B'—$C' \wedge A \neq B \Rightarrow CD \equiv C'D'$

$\exists ABC, \neg A$—B—$C \wedge \neg B$—C—$A \wedge \neg C$—A—B

$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$
A—B—$C \vee B$—C—$A \vee C$—A—B

A—D—$T \wedge B$—D—$C \wedge A \neq D \Rightarrow$
$\exists XY, A$—B—$X \wedge A$—C—$Y \wedge X$—T—Y

Syntactic proof

Identity for betweenness	$A—B—A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A—B—E \wedge BE \equiv CD$
Pasch	$A—P—C \wedge B—Q—C \Rightarrow \exists X, P—X—B \wedge Q—X—A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A—B—C \wedge A'—B'—C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A—B—C \wedge \neg B—C—A \wedge \neg C—A—B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A—B—C \vee B—C—A \vee C—A—B$
Euclid	$A—D—T \wedge B—D—C \wedge A \neq D \Rightarrow$ $\exists XY, A—B—X \wedge A—C—Y \wedge X—T—Y$

Syntactic proof

Identity for betweenness	$A—B—A \Rightarrow A = B$
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Lower 2-Dimensional	$\exists ABC, \neg A—B—C \wedge \neg B—C—A \wedge \neg C—A—B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A—B—C \vee B—C—A \vee C—A—B$
Euclid	$A—D—T \wedge B—D—C \wedge A \neq D \Rightarrow$ $\exists XY, A—B—X \wedge A—C—Y \wedge X—T—Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, X \in \Xi \wedge Y \in \Upsilon \Rightarrow A—X—Y)) \Rightarrow$ $\exists B, (\forall XY, X \in \Xi \wedge Y \in \Upsilon \Rightarrow X—B—Y)$
Point equality decidability	$X = Y \vee X \neq Y$

Parallel postulates are not *equivalent*

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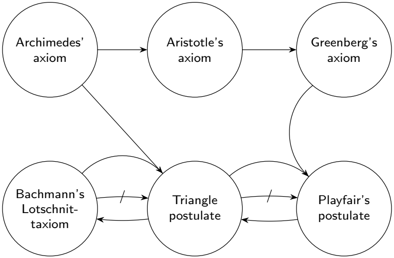

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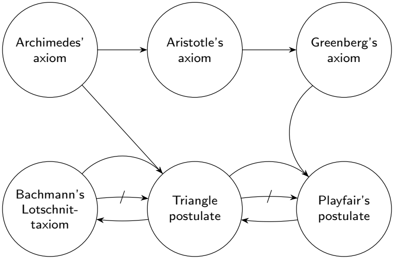

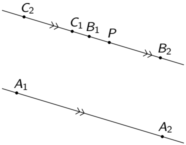

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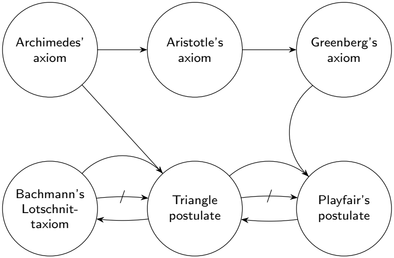

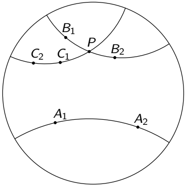

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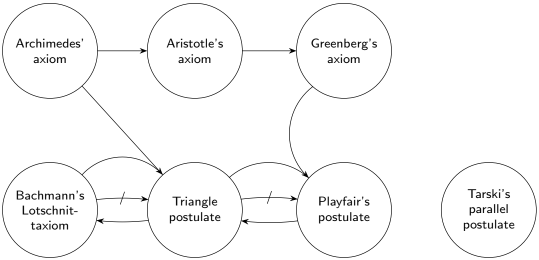

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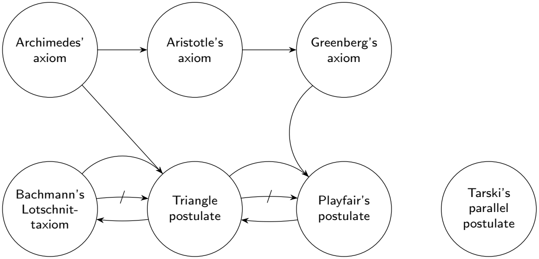

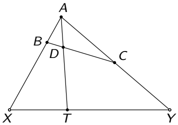

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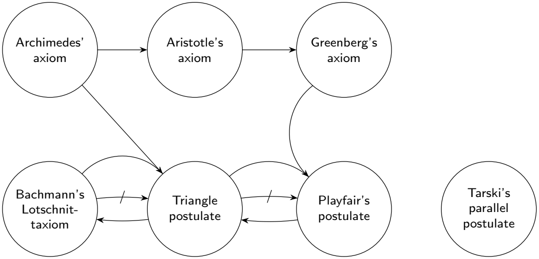

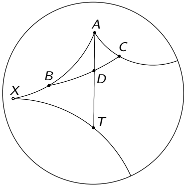

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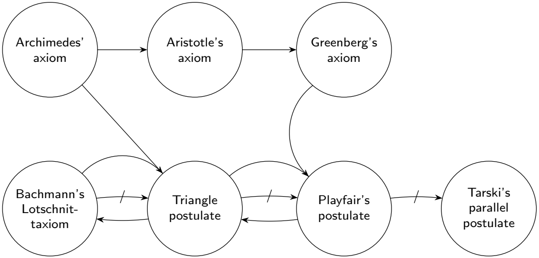

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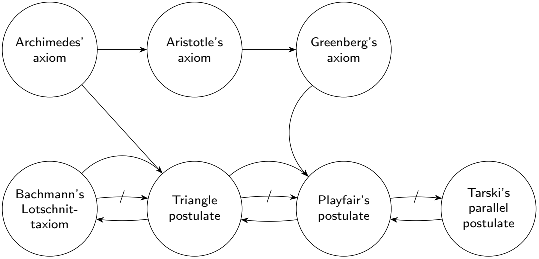

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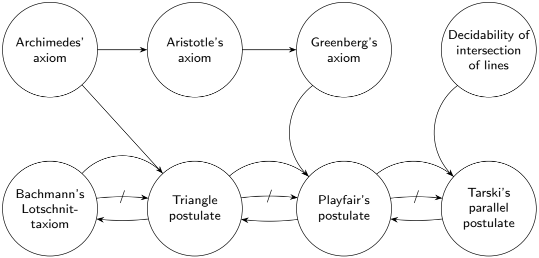

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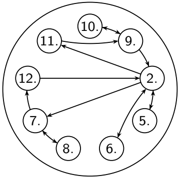

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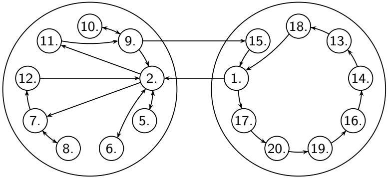

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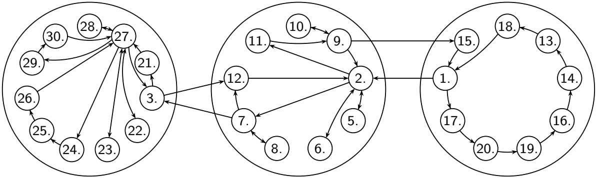

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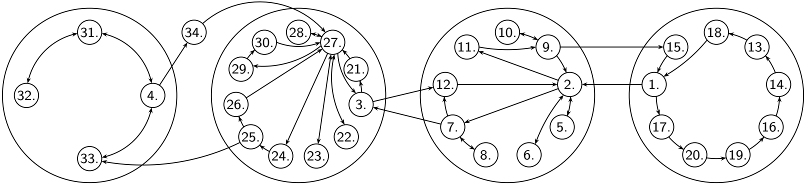

How to classify the postulates?

Pursuing the project faithfully will require that we take the extreme measure of shutting out the entreaties of our intuitions and imaginations - a forced separation of mental powers that will quite understandably be confusing and difficult to maintain [...].

(Richard J. Trudeau)

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Links between continuity axioms

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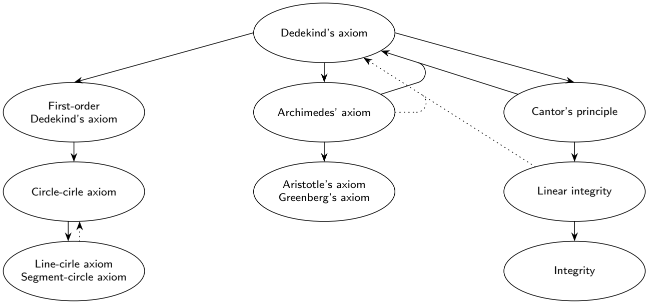

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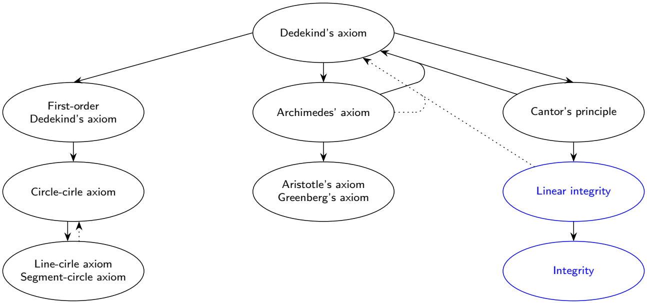

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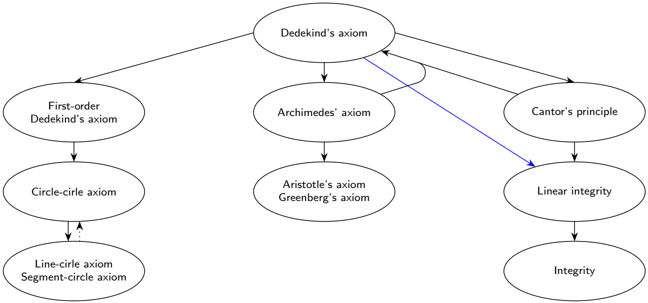

Formalized results about foundations of geometry

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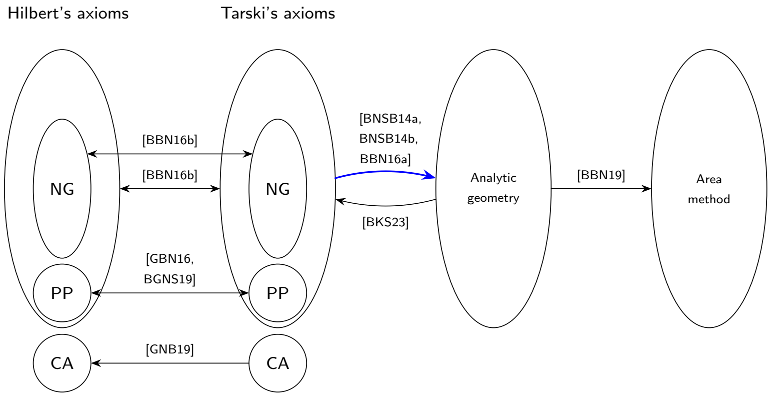
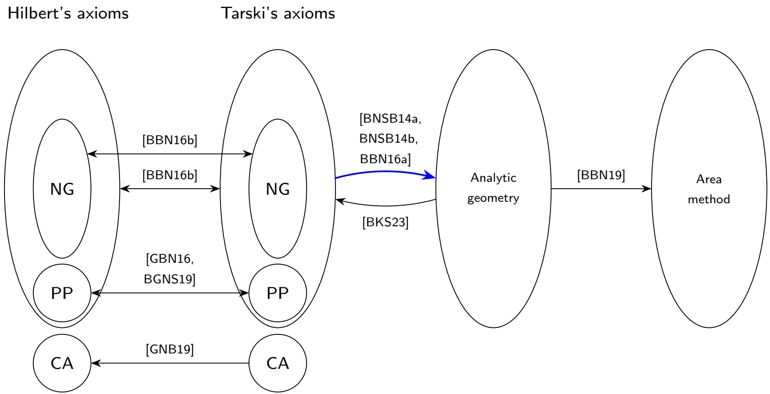


NG : neutral geometry, PP : parallel postulat, AC : continuity axiom.

Formalized results about foundations of geometry



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The automation developed in [BNSB14b, BBN19] was essential!

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- The power of computers can be leveraged.

Ongoing projects

This STSM

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- Work on the new rules introduced in *Different Types of Locus Dependencies in Solving Geometry Construction Problems*.
- Extend these tools to produce a trace allowing to construct a Rocq proof.

Upcoming ADG talks

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- In collaboration with Alexandre Jean and Nicolas Magaud: *An Automated Approach towards Constructivizing the GeoCoq Library*.

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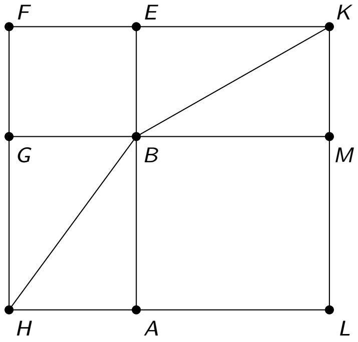
- In collaboration with Prunelle Colin: *On the Coq/Rocq Mechanization of Beeson's "On the Notion of Equal Figures in Euclid"*.

Beeson's "On the Notion of Equal Figures in Euclid"



Future Belgrade-Strasbourg collaborations?

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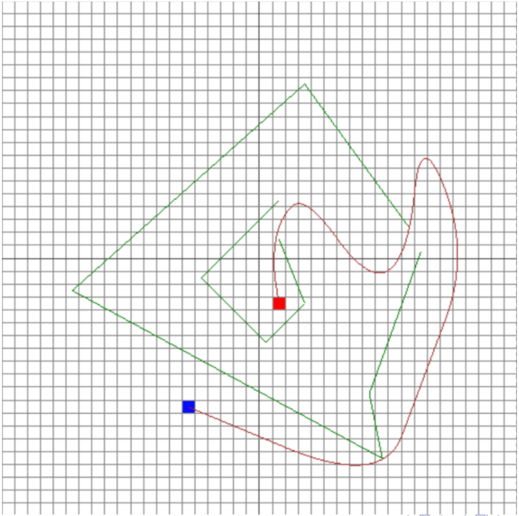
Safe smooth paths between straight line obstacles



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- Keep on working with the ARGO group!

Danijela Simić, Filip Marić, and Pierre Boutry.
Formalization of the Poincaré Disc Model of Hyperbolic Geometry.
Journal of Automated Reasoning, Volume 65, 2021.

Pierre Boutry, Gabriel Braun, and Julien Narboux.
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Towards an Independent Version of Tarski's System of Geometry.

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Charly Gries, Julien Narboux, and Pierre Boutry.

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In *Actes des Journées Francophones des Langages Applicatifs (JFLA 2019)*, Les Rousses, France, January 2019. Nicolas Magaud and Zaynah Dargaye.

Charly Gries, Pierre Boutry, and Julien Narboux.

Somme des angles d'un triangle et unicité de la parallèle : une preuve d'équivalence formalisée en Coq.

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Gabriel Braun, Pierre Boutry, and Julien Narboux.

From Hilbert to Tarski.

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Pierre Boutry, Julien Narboux, Pascal Schreck, and Gabriel Braun.

Using small scale automation to improve both accessibility and readability of formal proofs in geometry.

In *10th Int. Workshop on Automated Deduction in Geometry*, Coimbra, Portugal, July 2014.

Pierre Boutry, Julien Narboux, Pascal Schreck, and Gabriel Braun.

A short note about case distinctions in Tarski's geometry.

In *10th Int. Workshop on Automated Deduction in Geometry*, Coimbra, Portugal, July 2014.